Definite Integration with u-Substitution - Homework

Find the values of the following definite integrals. Verify using your calculator. Some will use *u*-substitution, others will not.

1. $\int_{-2}^{2}\left(x^3-1\right)dx$ $= \frac{x^4}{4} - x\Big]_{-2}^{2}$

$=\left(\frac{16}{4}-2\right)-\left(\frac{16}{4}+2\right)=2-6=\boxed{-4}$

2. $\int_{0}^{4} x\left(\sqrt{x}-1\right)dx$ $=\int_0^4 (x^{3/2}-x)dx$

$=\frac{2x^{5/2}}{5}-\frac{x^2}{2}\Big]_0^4=\left(\frac{2}{5}4^{5/2}-\frac{16}{2}\right)-0$

$\boxed{=\frac{64}{5}-8}$

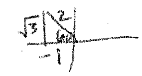
3. $\int_{0}^{\pi/3}\sin(2x)\,dx$

$u=2x \quad du=2dx$

$\frac{1}{2}du=dx$

$\frac{1}{2}\int_0^{2\pi/3}\sin(u)du=-\frac{1}{2}\cos(u)\Big]_0^{2\pi/3}=$

$-\frac{1}{2}\left(\cos\frac{2\pi}{3}-\cos(0)\right)=-\frac{1}{2}\left(-\frac{1}{2}-1\right)=\boxed{\frac{3}{4}}$



4. $\int_{0}^{\pi/12}\left(1-\cos 2x\right)dx$ =

$x-\frac{1}{2}\sin(2x)\Big]_0^{\pi/12}=\left(\frac{\pi}{12}-\frac{1}{2}\sin\left(2\cdot\frac{\pi}{12}\right)\right)-$

$0-\frac{1}{2}\sin(0)) = \left(\frac{\pi}{12}-\frac{1}{2}\sin\left(\frac{\pi}{6}\right)\right)-0$

$\frac{\pi}{12}-\frac{1}{2}\cdot\frac{1}{2}=\boxed{\frac{\pi}{12}-\frac{1}{4}}$

5. $\int_{0}^{1} 2x\left(x^2+1\right)^2dx$

$u=x^2+1$

$du=2xdx$

$u(0)=1 \quad u(1)=2$

$\int_1^2 u^2du=\frac{u^3}{3}\Big]_1^2=$

$\frac{8}{3}-\frac{1}{3}=\boxed{\frac{7}{3}}$

6. $\int_{0}^{3} x\sqrt{9-x^2}\,dx$

$u=9-x^2 \quad du=-2xdx$

$-\frac{1}{2}du=xdx$

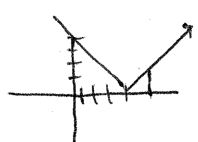
$u(0)=9 \quad u(3)=0$

$-\frac{1}{2}\int_9^0 u^{1/2}du=$

$-\frac{1}{\cancel{2}}\frac{\cancel{2}}{3}u^{3/2}\Big]_9^0=-\frac{1}{3}\left[0^{3/2}-9^{3/2}\right]=-\frac{1}{3}(0-27)$

$\boxed{=9}$

7. $\int_{0}^{5}\left|x-4\right|dx$

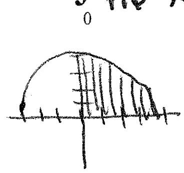


$\frac{1}{2}(4)(4)+\frac{1}{2}(1)(1)=8+\frac{1}{2}=\boxed{8.5}$

8. $\int_{0}^{4}\sqrt{16-x^2}\,dx$

semicircle w/ radius r=4

$A=\frac{\pi 4^2}{4}=\boxed{4\pi}$



9. $\int_{2}^{3}\frac{x}{\left(x^2-3\right)^2}dx$

$u=x^2-3$

$du=2x\,dx$

$\frac{1}{2}du=x\,dx$

$u(2)=1 \quad u(3)=6$

$\frac{1}{2}\int_1^6 u^{-2}du=\frac{1}{2}\frac{u^{-1}}{-1}=-\frac{1}{2u}\Big]_1^6=$

$-\frac{1}{2}\left[\frac{1}{6}-\frac{1}{1}\right]=-\frac{1}{2}\left[-\frac{5}{6}\right]=\boxed{\frac{5}{12}}$

10. $\int_{0}^{4}\frac{dt}{\sqrt{2t+1}}$

$u=2t+1$

$du=2dt$

$\frac{1}{2}du=dt$

$u(0)=1$

$u(4)=9$

$\frac{1}{2}\int_1^9 u^{-1/2}du$

$\frac{1}{\cancel{2}}\cancel{2}u^{1/2}\Big]_1^9=\left[9^{1/2}-1^{1/2}\right]$

$=3-1=\boxed{2}$

11. $\int_{0}^{\pi/2}\cos^3 t\sin t\,dt$

$u=\cos t$

$du=-\sin t\,dt$

$-du=\sin t\,dt$

$u(0)=\cos(0)=1$

$u(\pi/2)=\cos(\pi/2)=0$

$-\int_1^0 u^3du$

$=\int_0^1 u^3du=\frac{u^4}{4}\Big]_0^1=\frac{1}{4}-0=\boxed{\frac{1}{4}}$

12. $\int_{0}^{\sqrt{\pi/2}} t\sin\left(\pi-t^2\right)dt$

$u=\pi-t^2$

$du=-2t\,dt$

$-\frac{1}{2}du=t\,du$

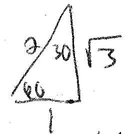
$u(0)=\pi$

$u(\sqrt{\pi/2})=\pi-\frac{\pi}{2}$

$-\frac{1}{2}\int_\pi^{\pi/2}\sin(u)du$

$+\frac{1}{2}\cos(u)\Big]_\pi^{\pi/2}=\frac{1}{2}\left[\cos\frac{\pi}{2}-\cos\pi\right]$

$=\frac{1}{2}(0--1)=\frac{1}{2}(1)=\boxed{\frac{1}{2}}$



13. $\int_0^{\pi/4} \sqrt{\tan x}\,\sec^2 x\,dx$ $\quad u=\tan x$, $du=\sec^2 x\,dx$, $u(0)=\tan 0=0$, $u(\pi/4)=\tan(\pi/4)=1$

$\Rightarrow \int_0^1 u^{1/2}\,du = \frac{2u^{3/2}}{3}\Big]_0^1 = \frac{2}{3}-0=\boxed{\frac{2}{3}}$

14. $\int_0^{\pi/3} \cos x\sqrt{1-\cos^2 x}\,dx =$

$\int_0^{\pi/3} \cos x\sqrt{\sin^2 x}\,dx =$

$\int_0^{\pi/3} \cos x \cdot \sin x\,dx$ $\quad u=\sin x$, $du=\cos x\,dx$, $u(0)=\sin(0)=0$, $u(\pi/3)=\sin(\pi/3)=\frac{\sqrt{3}}{2}$

$\int_0^{\sqrt{3}/2} u\,du = \frac{u^2}{2}\Big]_0^{\sqrt{3}/2}$

$= \frac{1}{2}\left[\left(\frac{\sqrt{3}}{2}\right)^2-0\right]=\boxed{\frac{3}{8}}$

15. $\int_{-1}^{1} e^{3x+2}\,dx$

$u=3x+2$ $\quad du=3dx$, $\frac{1}{3}du=dx$

$\frac{1}{3}\int_{-1}^{5} e^u\,du$ $\quad u(-1)=-3+2=-1$, $u(1)=3+2=5$

$\frac{1}{3}e^u\Big]_{-1}^{5} = \boxed{\frac{1}{3}\left(e^5-e^{-1}\right)}$

16. $\int_{-2}^{2} 2t^2+1\,dt = 2\int_0^2 (2t^2+1)\,dt =$

$2\left(\frac{2t^3}{3}+t\right]_0^2 = 2\left(\frac{2}{3}(8)+2\right)-0$

$=\frac{32}{3}+4=\boxed{\frac{44}{3}}$

17. $\int_{-4}^{4} 3x^5-4x^3\,dx = \boxed{0}$

since $3x^5-4x^3$ is odd!

18. $\int_{-4}^{4} \frac{1}{x^2}\,dx$ Explain why this doesn't exist.

$\frac{1}{x^2}$ has a vertical asymptote at $x=0$.

$f(x)$ needs to be continuous for us to use the FTC.

If $\int_0^2 f(x)\,dx=\frac{11}{3}$ and $\int_0^6 f(x)\,dx=15$, $f(x)$ is an even function (symmetric to the y - axis), find the following:

19. $\int_{-2}^{0} f(x)\,dx$ $\quad =\boxed{\frac{11}{3}}$

20. $\int_{-2}^{2} f(x)\,dx$ $\quad \boxed{\frac{22}{3}}$

21. $\int_0^2 -f(x)\,dx$ $\quad =\boxed{-\frac{11}{3}}$

22. $\int_{-2}^{0} 3f(x)\,dx$ $\quad =3\left(\frac{11}{3}\right)=\boxed{11}$

23. $\int_0^2 f(3x)\,dx$ $\quad u=3x$, $du=3dx$, $\frac{1}{3}du=dx$

$\frac{1}{3}\int_0^6 f(u)\,du=\frac{1}{3}(15)=\boxed{5}$

If $\int_0^2 f(x)\,dx=\frac{11}{3}$ and $\int_0^6 f(x)\,dx=15$, $f(x)$ is an odd function (symmetric to the origin), find the following:

24. $\int_{-2}^{0} f(x)\,dx$ $\quad =\boxed{-\frac{11}{3}}$

25. $\int_{-2}^{2} f(x)\,dx$ $\quad =\boxed{0}$

26. $\int_0^2 -f(x)\,dx$ $\quad \boxed{-\frac{11}{3}}$

27. $\int_{-2}^{0} 3f(x)\,dx$ $\quad 3\left(-\frac{11}{3}\right)=\boxed{-11}$

28. $\int_{-2}^{2} f(3x)\,dx$ $\quad u=3x$, $du=3dx$, $\frac{du}{3}=dx$

$\frac{1}{3}\int_{-6}^{6} f(u)\,du$

$\frac{1}{3}(0)=\boxed{0}$

Homework

Example 3: Given $\int_a^b f(x)dx = 5$ *and* $\int_a^b g(x)dx = -4$, use the properties to evaluate the following:

a. $\int_b^a f(x)dx$

$-\int_a^b f(x)dx = \boxed{-5}$

b. $\int_a^b 3f(x)dx$

$3\int_a^b f(x)dx$

$3(5) = \boxed{15}$

c. $\int_a^b 3dx$

$\boxed{3(b-a)}$

d. $\int_a^b (3+f(x))dx$

$\int_a^b 3dx + \int_a^b f(x)dx$

$\boxed{3(b-a)+5}$

e. $\int_a^b (f(x)+g(x))dx$

$\int_a^b f(x)dx + \int_a^b g(x)dx$

$5 + (-4) = \boxed{1}$

f. $\int_a^b (f(x)-g(x))dx$

$\int_a^b f(x)dx - \int_a^b g(x)dx$

$5 - (-4)$

$\boxed{9}$

g. $\int_a^6 g(x)dx + \int_6^b g(x)dx$

$\int_a^b g(x)dx$

$\boxed{-4}$

4. If $\int_2^5 (2f(x)+3)dx = 17$, find $\int_2^5 f(x)dx$

$\int_2^5 2f(x)dx + \int_2^5 3dx = 17 \mid 2\int_2^5 f(x)dx + 9 = 17 \mid 2\int_2^5 f(x)dx = 8$

$\boxed{\int_2^5 f(x)dx = 4}$

20. $\int_1^2 (4x^3 - 6x)dx = x^4 - 3x^2\Big|_1^2 = (16-12) - (1-3)$

$4 + 2 = 6$

A. 2 B. 4 C. 6 D. 36 E. 42

$\theta = \frac{\pi}{2}$ $u = 2$

$\theta = 0$ $u = 1$

$u = 1 + \sin(\theta)$

$\frac{du}{d\theta} = \cos(\theta)$

$\frac{1}{\cos(\theta)} du = d\theta$

12. $\int_0^{\frac{\pi}{2}} \frac{\cos\theta}{\sqrt{1+\sin\theta}} d\theta = \int_1^2 \frac{\cos(\theta)}{\sqrt{u}} d\theta = \int_1^2 \frac{\cancel{\cos(\theta)}}{u^{1/2}} \frac{1}{\cancel{\cos(\theta)}} du = \int_1^2 u^{-1/2} du = 2u^{1/2}\Big|_1^2$

$2\sqrt{2} - 2\sqrt{1}$

$2\sqrt{2} - 2 = 2(\sqrt{2} - 1)$

A. $-2(\sqrt{2}-1)$ B. $-2\sqrt{2}$ C. $2\sqrt{2}$ D. $2(\sqrt{2}-1)$ E. $2(\sqrt{2}+1)$

you could u-sub too

13. $\int_0^1 (3x-2)^2 dx = \int_0^1 9x^2 - 12x + 4\, dx = 3x^3 - 6x^2 + 4x\Big|_0^1 = (3-6+4) - (0-0+0)$

1

A. $-\frac{7}{3}$ B. $-\frac{7}{9}$ C. $\frac{1}{9}$ D. 1 E. 3

$x = 3$ $u = 10$

$x = 2$ $u = 5$

$u = x^2 + 1$

$\frac{du}{dx} = 2x$

$\frac{1}{2x} du = dx$

14. $\int_2^3 \frac{x}{x^2+1} dx = \int_5^{10} \frac{x}{u} dx = \int_5^{10} \frac{x}{u} \frac{1}{2x} du = \int_5^{10} \frac{1}{2} \cdot \frac{1}{u} du = \frac{1}{2} \ln|u| \Big|_5^{10} = \frac{1}{2}\ln(10) - \frac{1}{2}\ln(5)$

$\frac{1}{2}[\ln(10) - \ln(5)]$

$\frac{1}{2}\ln(10/5) = \frac{1}{2}\ln(2)$

A. $\frac{1}{2}\ln\frac{3}{2}$ B. $\frac{1}{2}\ln 2$ C. $\ln 2$ D. $2\ln 2$ E. $\frac{1}{2}\ln 5$

On this worksheet you will use substitution, as well as the other integration rules, to evaluate the the given definite and indefinite integrals.

Steps for integration by Substitution

1. Determine u: think parentheses and denominators
2. Find $\frac{du}{dx}$
3. Rearrange $\frac{du}{dx}$ until you can make a substitution
4. Make the substitution to obtain an integral in u
5. Integrate with respect to u
6. Substitute u back to be left with an expression in terms of x

Steps for finding the Definite Integral

1. Using substitution or otherwise, find an antiderivative $F(x)$
2. Using the given limits of integration, find $F(b) - F(a)$. Remember: b is the upper limit and a is the lower limit.

   Be careful to evaluate $-F(a)$ correctly (distribute the negative accordingly)

   Your answer should be a *number*

   If you make a substitution, remember to substitute back before plugging in your limits of integration

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Example 1:

Find $\int 4x\left(x^2+1\right))^5\,dx$.

Observe that if $u = x^2+1$ then $\dfrac{du}{dx} = 2x$ and so

$$du = 2x\,dx \Longrightarrow 2\,du = 4x\,dx$$

$$\Longrightarrow \int 4x\left(x^2+1\right)^5\,dx = \int 2u^5\,du = \frac{1}{3}u^6 + C = \frac{1}{3}(x^2+1)^6 + C.$$

So $\int 4x\left(x^2+1\right)^5\,dx = \dfrac{1}{3}(x^2+1)^6 + C$.

Example 2:

Find $\int_1^5 9x^2 + 10x + 3\,dx$.

$$\int_1^5 9x^2 + 10x + 3\,dx = 3x^3 + 5x^2 + 3x\Big|_1^5 = 3(5)^3 + 5(5)^2 + 3(5) - \left[3(1)^3 + 5(1)^2 + 3(1)\right] = 515 - 11 = 504$$

So $\int_1^5 9x^2 + 10x + 3\,dx = 504$.

Example 3:

Find $\int_0^2 \dfrac{2x}{\sqrt{x^2+4}}\,dx$.

Observe that if $u = x^2+4$ then $\dfrac{du}{dx} = 2x$ and so

$$du = 2x\,dx$$

$$\Longrightarrow \int_0^2 \frac{2x}{\sqrt{x^2+4}}\,dx = \int_{x=0}^{x=2} u^{-1/2}\,du = 2u^{1/2}\Big|_{x=0}^{x=2} = 2\left(x^2+4\right)^{1/2}\Big|_0^2 = 2\left(2^2+4\right)^{1/2} - 2\left(0^2+4\right)^{1/2} = 2\sqrt{8} - 4 = 4\left(\sqrt{2}-1\right).$$

So $\int_0^2 \dfrac{2x}{\sqrt{x^2+4}}\,dx = 4\left(\sqrt{2}-1\right)$.

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1. $\int_1^2 \frac{1}{r^2}\,dr$
2. $\int \frac{3}{3x+5}\,dx$
3. $\int (4x+1)^8\ dx$
4. $\int_0^6 6y\,dy$
5. $\int 4x\left(2x^2+1\right)^5\ dx$
6. $\int_9^3 t^3\,dt$
7. $\int_0^1 \frac{3s^2+2}{2s^3+4s+3}\,ds$
8. $\int \frac{3x^2}{x^3+8}\,dx$
9. $\int_{-2}^7 12s^2+1\,ds$
10. $\int_0^1 (2x-1)^6\,dx$
11. $\int \frac{z}{(2-z)(2+z)}\,dz$
12. $\int 4te^{t^2}\,dt$
13. $\int \frac{6x}{3x^2+5}\,dx$
14. $\int_0^2 3y\sqrt{4-y^2}\,dy$
15. $\int \frac{x}{2x^2+3}\,dx$
16. $\int_0^1 6x^2e^{x^3}\,dx$
17. $\int \frac{6t}{2t^2+5}\,dt$
18. $\int \frac{e^{3x}}{4-e^{3x}}\,dx$
19. $\int_{-\ln(2)}^0 \frac{2e^x}{e^x+1}\,dx$
20. $\int \frac{3}{x\ln(x)}\,dx$ *
21. $\int_1^4 \frac{1}{2\sqrt{x}\left(1+\sqrt{x}\right)^2}\,dx$ *
22. $\int_1^2 x(x-1)^4\,dx$ *
23. $\int \frac{2x}{(4x+5)^3}\,dx$ *

Answers

1. Ans. 0.5
2. $u=3x+5$, Ans. $\ln|3x+5|+C$
3. $u=4x+1)^8$, Ans. $\frac{1}{36}(4x+1)^9$
4. Ans. 108
5. $u=2x^2+1$, Ans. $\frac{1}{6}(2x^2+1)^6+C$
6. Ans. -1620
7. $u=2s^3+4s+3$, Ans. $\frac{\ln(3)}{2}$
8. $u=x^3+8$, Ans. $\ln|x^3+8|+C$
9. Ans. 1413
10. $u=2x-1$, Ans. $\frac{1}{7}$
11. $u=4-z^2$, Ans. $-\frac{1}{2}\ln|4-z^2|+C$
12. $u=t^2$, Ans. $2e^{t^2}+C$
13. $u=3x^2+5$, Ans. $\ln(3x^2+5)+C$
14. $u=4-y^2$, Ans. 8
15. $u=2x^2+3$, Ans. $\frac{1}{4}\ln(2x^2+3)+C$
16. $u=x^3$, Ans. $2e-2$
17. $u=2t^2+5$, Ans. $\frac{3}{2}\ln(2t^2+5)+C$
18. $u=4-e^{3x}$, Ans. $-\frac{1}{3}\ln|4-e^{3x}|+C$
19. $u=e^x+1$, Ans. $2\ln\left(\frac{4}{3}\right)$
20. $u=\ln(x)$, Ans. $3\ln|\ln|x||+C$
21. $u=1+\sqrt{x}$, Ans. $\frac{1}{6}$
22. $u=x-1$, Ans. $\frac{11}{30}$
23. $u=4x+5$, Ans. $-\frac{8x+5}{16(4x+5)^2}+C$

Selected Worked Answers
6.6
Final Set of Integrals

1. $\int_1^2 \frac{1}{r^2}dr = \int_1^2 r^{-2}dr = -r^{-1}\Big|_1^2 = (-2^{-1})-(-1^{-1}) = -\frac{1}{2}+1 = \frac{1}{2}$

No u-sub needed (not all problems use u-substitution: e.g. #4, 6, 9, etc)

2. $\int \frac{3}{3x+5}dx$ $\quad u=3x+5$, $\frac{du}{dx}=3$, $\frac{1}{3}du = dx$

Indefinite integral: needs +C

$\int \frac{3}{u}\frac{1}{3}du$

$\int \frac{1}{u}du = \ln|u|+C = \ln|3x+5|+C$

7. $\int_0^1 \frac{3s^2+2}{2s^3+4s+3}ds$ $\quad u=2s^3+4s+3$, $\frac{du}{ds}=6s^2+4$, $\frac{1}{6s^2+4}du = ds$ $\quad s=1 \to u=9$, $s=0 \to u=3$

$\int_3^9 \frac{3s^2+2}{u}\frac{1}{6s^2+4}du$

$\int_3^9 \frac{1}{u}\cdot\frac{1}{2}du = \int_3^9 \frac{1}{2}\frac{1}{u}du = \frac{1}{2}\ln|u|\Big|_3^9 = \frac{1}{2}\ln|9| - \frac{1}{2}\ln|3| = \frac{1}{2}\ln(9/3) = \frac{\ln(3)}{2}$

11. $\int \frac{z}{(2-z)(2+z)}dz = \int \frac{z}{4-z^2}dz$ $\quad u=4-z^2$, $\frac{du}{dz}=-2z$, $-\frac{1}{2z}du = dz$

$\int \frac{z}{u}\left(\frac{-1}{2z}\right)du$

$\int \frac{1}{u}\left(-\frac{1}{2}\right)du = \int -\frac{1}{2}\frac{1}{u}du = -\frac{1}{2}\ln|u|+C = -\frac{1}{2}\ln|4-z^2|+C$

16. $\int_0^1 6x^2e^{x^3}dx$ $\quad u=x^3$, $\frac{du}{dx}=3x^2$, $\frac{1}{3x^2}du=dx$ $\quad x=1\to u=1$, $x=0\to u=0$

$\int_0^1 6x^2e^u\frac{1}{3x^2}du$

$\int_0^1 2e^u du = 2e^u\Big|_0^1 = 2e^1-2e^0 = 2e-2$

21. $\int_1^4 \frac{1}{2\sqrt{x}(1+\sqrt{x})^2}dx$ $\quad u=1+\sqrt{x}$, $\frac{du}{dx}=\frac{1}{2}x^{-1/2}=\frac{1}{2\sqrt{x}}$, $2\sqrt{x}\,du=dx$ $\quad x=4\to u=3$, $x=1\to u=2$

$\int_2^3 \frac{1}{2\sqrt{x}\,u^2}2\sqrt{x}\,du$

$\int_2^3 \frac{1}{u^2}du = \int_2^3 u^{-2}du = -u^{-1}\Big|_2^3 = -\frac{1}{3}+\frac{1}{2} = \frac{1}{6}$