

6.7 Integration Using Long Division and Completing the Square

Given the integral looks like the derivative of an inverse trig function...

(a is a constant and u is a function)

$$I. \int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin\left(\frac{u}{a}\right) + c$$

$$III. \int \frac{du}{a^2 + u^2} = \frac{1}{a} \arctan\left(\frac{u}{a}\right) + c$$

$$II. \int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \operatorname{arcsec}\left(\frac{|u|}{a}\right) + c$$

$$\text{Example: } \int \frac{4}{x^2+5} dx = 4 \int \frac{1}{x^2+5} dx = 4\left(\frac{1}{\sqrt{5}}\right) \tan^{-1}\left(\frac{x}{\sqrt{5}}\right)$$

How do you deal with integrals of rational functions?

1. U-Substitution
2. Long Division (Polynomial Long Division)
3. Completing the Square
4. Linear Partial Fractions

Technique 2: Polynomial Long Division: Useful when _____

$$\int \frac{x^3 + 3x^2 + 8x + 19}{x^2 + 5} dx =$$

$$\begin{array}{r} x+3 \\ x+5 \overline{) x^3 + 3x^2 + 8x + 19} \\ \underline{-(x^3 + 5x)} \\ 3x^2 + 3x + 19 \\ \underline{-(3x^2 + 15)} \\ 3x + 4 \end{array}$$

$$\int x+3 + \frac{3x+4}{x^2+5} dx$$

$$\int x+3 dx + \int \frac{3x+4}{x^2+5} dx$$

$$\int x+3 dx + \int \frac{3x}{x^2+5} dx + \int \frac{4}{x^2+5} dx$$

$$\begin{aligned} u &= x^2+5 \\ \frac{du}{dx} &= 2x \\ \frac{1}{2x} du &= dx \end{aligned}$$

$$\int x+3 dx + \int \frac{3}{2} \frac{1}{u} du + \int \frac{4}{x^2+5} dx$$

$$\int x+3 dx + \int \frac{3}{2} \frac{1}{u} du + 4 \int \frac{1}{x^2+5} dx$$

$$\frac{x^2}{2} + 3x + \frac{3}{2} \ln|x^2+5| + 4\left(\frac{1}{\sqrt{5}}\right) \tan^{-1}\left(\frac{x}{\sqrt{5}}\right) + c$$

Technique 3: Completing the Square: Useful when

$$\int \frac{1}{x^2 - 4x + 8} dx = \int \frac{1}{(x-2)^2 + 4} dx = \int \frac{1}{(x-2)^2 + 2^2} dx = \frac{1}{4} \int \frac{1}{\left(\frac{x-2}{2}\right)^2 + 1} dx$$

$$= \frac{1}{4} \int \frac{1}{\left(\frac{1}{2}x - 1\right)^2 + 1} dx \quad u = \frac{1}{2}x - 1$$

$$\frac{du}{dx} = \frac{1}{2}$$

$$= \frac{1}{4} \int \frac{1}{u^2 + 1} 2 du \quad 2 du = dx$$

$$= \frac{1}{2} \int \frac{1}{u^2 + 1} du = \frac{1}{2} \tan^{-1}\left(\frac{1}{2}x - 1\right) + C$$

Practice:

$$\int \frac{6x^2 - 4x - 25}{x - 2} dx =$$

A $3x^2 + 8x - 9 \ln |x - 2| + C$

B $3x^2 + 8x + \frac{9}{(x-2)^2} + C$

C $(2x^3 - 2x^2 - 25x) \ln |x - 2| + C$

D $\frac{2x^3 - 2x^2 - 25x}{\frac{x^2}{2} - 2x} + C$

$$\int 6x + 8 dx + \int \frac{9}{x-2} dx$$

$$3x^2 + 8x + 9 \ln |x-2| + C$$

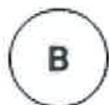
~~$$\begin{array}{r} 6x - 16 \\ x-2 \overline{) 6x^2 - 4x - 25} \\ \underline{-(6x^2 - 12x)} \\ 8x - 25 \\ \underline{-(8x - 16)} \\ 9 \end{array}$$~~

$$\begin{array}{r} 6x + 8 \\ x-2 \overline{) 6x^2 - 4x - 25} \\ \underline{-(6x^2 - 12x)} \\ 8x - 25 \\ \underline{-(8x - 16)} \\ 9 \end{array}$$

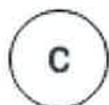
$$\int \frac{1}{x^2 + 4x + 5} dx = \int \frac{1}{(x+2)^2 + 1} dx = \tan^{-1}(x+2) + C$$



$$\arctan(x+2) + C$$



$$\arcsin(x+2) + C$$



$$\ln|x^2 + 4x + 5| + C$$



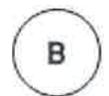
$$\frac{1}{\frac{1}{3}x^3 + 2x^2 + 5x} + C$$

$$\int_1^2 \frac{x^2 - x - 5}{x+2} dx =$$

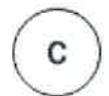
$$\begin{array}{r} x-3 \\ x+2 \overline{) x^2 - x - 5} \\ \underline{-(x^2 + 2x)} \\ -3x - 5 \\ \underline{-(-3x - 6)} \\ 1 \end{array}$$



$$-\frac{3}{2} + \ln \frac{4}{3}$$



$$-\frac{25}{21}$$



$$\frac{5}{2} + 3 \ln \frac{3}{4}$$



$$\frac{23}{45}$$

$$\int_1^2 x-3 dx + \int_1^2 \frac{1}{x+2} dx$$

$$\left(\frac{1}{2}x^2 - 3x\right)\Big|_1^2 + \ln|x+2|\Big|_1^2$$

$$(2-6) - \left(\frac{1}{2}-3\right) + \ln(4) - \ln(3)$$

$$-4 - \left(-\frac{5}{2}\right) + \ln\left(\frac{4}{3}\right)$$

$$-\frac{3}{2} + \ln\left(\frac{4}{3}\right)$$

How can I do U-Substitution fast?

Integrate $\int \cos(5x) dx$ and $\int e^{2x} dx$

$$\int \cos(5x) dx = \frac{1}{5} \sin(5x) + C$$

$$\int e^{2x} dx = \frac{1}{2} e^{2x} + C$$

We are undoing the derivative, So we can divide by the derivative of the inside.

Integrate the following:

$$\int \sin(7x) dx$$

$$-\frac{1}{7} \cos(7x) + C$$

$$\int \sec^2(2x) dx$$

$$\frac{1}{2} \tan(2x) + C$$

$$\int -\sin(9x) dx$$

$$\frac{1}{9} \cos(9x) + C$$

$$\int \cos\left(\frac{1}{3}x\right) dx$$

$$3 \sin\left(\frac{1}{3}x\right) + C$$

$$\int -\csc^2\left(\frac{1}{5}x\right) dx$$

$$5 \cot\left(\frac{1}{5}x\right) + C$$

$$\int -\cos\left(\frac{3}{7}x\right) dx$$

$$-\frac{7}{3} \sin\left(\frac{3}{7}x\right) + C$$

$$\int e^{4x} dx$$

$$\frac{1}{4} e^{4x} + C$$

$$\int e^{\frac{1}{2}x} dx$$

$$2e^{\frac{1}{2}x} + C$$

$$\int \frac{1}{3x+1} dx$$

$$\frac{1}{3} \ln|3x+1| + C$$

$$\int e^{5x} dx$$

$$\frac{1}{5} e^{5x} + C$$

$$\int \frac{1}{\frac{1}{2}x-2} dx$$

$$2 \ln|\frac{1}{2}x-2| + C$$

$$\int -\cos(6x) dx$$

$$-\frac{1}{6} \sin(6x) + C$$