6.7 Integration Using Long Division and Completing the Square

Given the integral looks like the derivative of an inverse trig function…
(a is a constant and u is a function)

I. $\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\left(\frac{u}{a}\right) + c$

II. $\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\text{arcsec}\left(\frac{|u|}{a}\right) + c$

III. $\int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\left(\frac{u}{a}\right) + c$

Example: $\int \frac{4}{x^2+5}dx = 4\int \frac{1}{x^2+5}dx = 4\left(\frac{1}{\sqrt{5}}\right)\tan^{-1}\left(\frac{x}{\sqrt{5}}\right)$

How do you deal with integrals of rational functions?

1. U-Substitution
2. Long Division (Polynomial Long Division)
3. Completing the Square
4. Linear Partial Fractions

Technique 2: Polynomial Long Division: Useful when ______________________________

$$\int \frac{x^3+3x^2+8x+19}{x^2+5}dx =$$

$$\begin{array}{r|l} & x \quad +3 \\ \hline x^2+5 & x^3+3x^2+8x+19 \\ & -(x^3 \qquad +5x) \\ \hline & 3x^2+3x+19 \\ & -(3x^2 \qquad +15) \\ \hline & 3x+4 \end{array}$$

$\int x+3 + \frac{3x+4}{x^2+5}dx$

$\int x+3dx + \int \frac{3x+4}{x^2+5}dx$

$\int x+3dx + \int \frac{3x}{x^2+5}dx + \int \frac{4}{x^2+5}dx$

$u = x^2+5$

$\frac{du}{dx} = 2x$

$\frac{1}{2x}du = dy$

$\int x+3dx + \int \frac{3}{2}\frac{1}{u}du + \int \frac{4}{x^2+5}dx$

$\int x+3dx + \int \frac{3}{2}\frac{1}{u}du + 4\int \frac{1}{x^2+5}dx$

$\frac{x^2}{2} + 3x + \frac{3}{2}\ln|x^2+5| + 4\left(\frac{1}{\sqrt{5}}\right)\tan^{-1}\left(\frac{x}{\sqrt{5}}\right) + C$

Technique 3: Completing the Square: Useful when ______________________________

$$\int \frac{1}{x^2-4x+8}dx = \int \frac{1}{(x-2)^2+4}dx = \int \frac{1}{(x-2)^2+2^2}dx = \frac{1}{4}\int \frac{1}{\left(\frac{x-2}{2}\right)^2+1}dx$$

$$= \frac{1}{4}\int \frac{1}{(\frac{1}{2}x-1)^2+1}dx$$

$u = \frac{1}{2}x-1$

$\frac{du}{dx} = \frac{1}{2}$

$2du = dx$

$$= \frac{1}{4}\int \frac{1}{u^2+1}2du$$

$$= \frac{1}{2}\int \frac{1}{u^2+1}du = \frac{1}{2}\tan^{-1}\left(\frac{1}{2}x-1\right)+C$$

Practice:

$$\int \frac{6x^2-4x-25}{x-2}dx =$$

[✓ A] $3x^2 + 8x - 9ln\ |x-2| + C$

B $3x^2 + 8x + \frac{9}{(x-2)^2} + C$

C $(2x^3 - 2x^2 - 25x)\ln|x-2| + C$

D $\frac{2x^3-2x^2-25x}{\frac{x^2}{2}-2x} + C$

~~$6x-16$~~

~~$x-2\ \overline{)\,6x^2-4x-25}$~~

~~$-(6x^2-12x)$~~

~~$-8x-25$~~

~~$-(-16x+32)$~~

$6x+8$

$x-2\ \overline{)\,6x^2-4x-25}$

$-(6x^2-12x)$

$8x-25$

$-(8x-16)$

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$\int 6x+8\,dx + \int \frac{9}{x-2}dx$

$3x^2+8x+9\ln|x-2|+C$

$$\int \frac{1}{x^2+4x+5}\,dx = \int \frac{1}{(x+2)^2+1}\,dx = \tan^{-1}(x+2)+C$$

[✓ A] $arctan\,(x+2)\ +\ C$

B $arcsin\,(x+2)\ +\ C$

C $ln\ |x^2+4x+5|\ +C$

D $\frac{1}{\frac{1}{3}x^3+2x^2+5x}+C$

$$\int_1^2 \frac{x^2-x-5}{x+2}\,dx =$$

$$\begin{array}{r|l} & x-3 \\ \hline x+2 & x^2-x-5 \\ & -(x^2+2x) \\ \hline & -3x-5 \\ & -(-3x-6) \\ \hline & 1 \end{array}$$

$$\int_1^2 x-3\,dx + \int_1^2 \frac{1}{x+2}\,dx$$

$$\left(\tfrac{1}{2}x^2-3x\right)\Big|_1^2 + \ln|x+2|\,\Big|_1^2$$

$$(2-6)-\left(\tfrac{1}{2}-3\right)+\ln(4)-\ln(3)$$

$$-4-\left(-\tfrac{5}{2}\right)\ +\ \ln(4/3)$$

$$-\tfrac{3}{2}+\ln(4/3)$$

[✓ A] $-\frac{3}{2}+\ln\frac{4}{3}$

B $-\frac{25}{21}$

C $\frac{5}{2}+3\ln\frac{3}{4}$

D $\frac{23}{45}$

How can I do U-Substitution fast?

Integrate $\int \cos(5x)\,dx$ and $\int e^{2x}\,dx$

$\int \cos(5x)dx = \frac{1}{5}\sin(5x)+C$

$\int e^{2x}dx = \frac{1}{2}e^{2x}+C$

We are undoing the derivative. So we can divide by the derivative of the inside.

Integrate the following:

$\int \sin(7x)\,dx$

$-\frac{1}{7}\cos(7x)+C$

$\int \sec^2(2x)\,dx$

$\frac{1}{2}\tan(2x)+C$

$\int -\sin(9x)dx$

$\frac{1}{9}\cos(9x)+C$

$\int \cos\left(\frac{1}{3}x\right)dx$

$3\sin(\frac{1}{3}x)+C$

$\int -\csc^2\left(\frac{1}{5}x\right)dx$

$5\cot(\frac{1}{5}x)+C$

$\int -\cos(\frac{3}{7}x)dx$

$-\frac{7}{3}\sin(\frac{3}{7}x)+C$

$\int e^{4x}dx$

$\frac{1}{4}e^{4x}+C$

$\int e^{\frac{1}{2}x}dx$

$2e^{\frac{1}{2}x}+C$

$\int \frac{1}{3x+1}dx$

$\frac{1}{3}\ln|3x+1|+C$

$\int e^{5x}dx$

$\frac{1}{5}e^{5x}+C$

$\int \frac{1}{\frac{1}{2}x-2}dx$

$2\ln|\frac{1}{2}x-2|+C$

$\int -\cos(6x)dx$

$-\frac{1}{6}\sin(6x)+C$