Question 6

(a) $f(0) = 3$ and $f'(0) = -2$

The third-degree Taylor polynomial for f about $x = 0$ is

$$3 - 2x + \frac{3}{2!}x^2 + \frac{-\frac{23}{2}}{3!}x^3 = 3 - 2x + \frac{3}{2}x^2 - \frac{23}{12}x^3.$$

$$2 : \begin{cases} 1 : \text{two terms} \\ 1 : \text{remaining terms} \end{cases}$$

(b) The first three nonzero terms of the Maclaurin series for e^x are $1 + x + \frac{1}{2!}x^2$.

The second-degree Taylor polynomial for $e^x f(x)$ about $x = 0$ is

$$\begin{aligned} &3\left(1 + x + \frac{1}{2!}x^2\right) - 2x(1 + x) + \frac{3}{2}x^2(1) \\ &\quad = 3 + (3 - 2)x + \left(\frac{3}{2} - 2 + \frac{3}{2}\right)x^2 \\ &\quad = 3 + x + x^2. \end{aligned}$$

$$2 : \begin{cases} 1 : \text{three terms for } e^x \\ 1 : \text{three terms for } e^x f(x) \end{cases}$$

(c)
$$\begin{aligned} h(1) &= \int_0^1 f(t)\,dt \\ &\approx \int_0^1 \left(3 - 2t + \frac{3}{2}t^2 - \frac{23}{12}t^3\right) dt \\ &= \left[3t - t^2 + \frac{1}{2}t^3 - \frac{23}{48}t^4\right]_{t=0}^{t=1} \\ &= 3 - 1 + \frac{1}{2} - \frac{23}{48} = \frac{97}{48} \end{aligned}$$

$$2 : \begin{cases} 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$$

(d) The alternating series error bound is the absolute value of the first omitted term of the series for $h(1)$.

$$\int_0^1 \left(\frac{54}{4!}t^4\right) dt = \left[\frac{9}{20}t^5\right]_{t=0}^{t=1} = \frac{9}{20}$$

$$\text{Error} \le \left|\frac{9}{20}\right| = 0.45$$

$$3 : \begin{cases} 1 : \text{uses fourth-degree term of Maclaurin series for } f \\ 1 : \text{uses first omitted term of series for } h(1) \\ 1 : \text{error bound} \end{cases}$$

Question 6

A function f has derivatives of all orders at $x = 0$. Let $P_n(x)$ denote the nth-degree Taylor polynomial for f about $x = 0$.

(a) It is known that $f(0) = -4$ and that $P_1\left(\frac{1}{2}\right) = -3$. Show that $f'(0) = 2$.

(b) It is known that $f''(0) = -\frac{2}{3}$ and $f'''(0) = \frac{1}{3}$. Find $P_3(x)$.

(c) The function h has first derivative given by $h'(x) = f(2x)$. It is known that $h(0) = 7$. Find the third-degree Taylor polynomial for h about $x = 0$.

(a) $P_1(x) = f(0) + f'(0)x = -4 + f'(0)x$

$$P_1\left(\frac{1}{2}\right) = -4 + f'(0)\cdot\frac{1}{2} = -3$$
$$f'(0)\cdot\frac{1}{2} = 1$$
$$f'(0) = 2$$

$$2 : \begin{cases} 1 : \text{uses } P_1(x) \\ 1 : \text{verifies } f'(0) = 2 \end{cases}$$

(b) $$P_3(x) = -4 + 2x + \left(-\frac{2}{3}\right)\cdot\frac{x^2}{2!} + \frac{1}{3}\cdot\frac{x^3}{3!}$$
$$= -4 + 2x - \frac{1}{3}x^2 + \frac{1}{18}x^3$$

$$3 : \begin{cases} 1 : \text{first two terms} \\ 1 : \text{third term} \\ 1 : \text{fourth term} \end{cases}$$

(c) Let $Q_n(x)$ denote the Taylor polynomial of degree n for h about $x = 0$.

$$h'(x) = f(2x) \Rightarrow Q_3'(x) = -4 + 2(2x) - \frac{1}{3}(2x)^2$$
$$Q_3(x) = -4x + 4\cdot\frac{x^2}{2} - \frac{4}{3}\cdot\frac{x^3}{3} + C;\ C = Q_3(0) = h(0) = 7$$
$$Q_3(x) = 7 - 4x + 2x^2 - \frac{4}{9}x^3$$

OR

$$h'(x) = f(2x),\ h''(x) = 2f'(2x),\ h'''(x) = 4f''(2x)$$
$$h'(0) = f(0) = -4,\ h''(0) = 2f'(0) = 4,\ h'''(0) = 4f''(0) = -\frac{8}{3}$$
$$Q_3(x) = 7 - 4x + 4\cdot\frac{x^2}{2!} - \frac{8}{3}\cdot\frac{x^3}{3!} = 7 - 4x + 2x^2 - \frac{4}{9}x^3$$

$$4 : \begin{cases} 2 : \text{applies } h'(x) = f(2x) \\ 1 : \text{constant term} \\ 1 : \text{remaining terms} \end{cases}$$

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Question 3

x	$h(x)$	$h'(x)$	$h''(x)$	$h'''(x)$	$h^{(4)}(x)$
1	11	30	42	99	18
2	80	128	$\frac{488}{3}$	$\frac{448}{3}$	$\frac{584}{9}$
3	317	$\frac{753}{2}$	$\frac{1383}{4}$	$\frac{3483}{16}$	$\frac{1125}{16}$

Let h be a function having derivatives of all orders for $x > 0$. Selected values of h and its first four derivatives are indicated in the table above. The function h and these four derivatives are increasing on the interval $1 \le x \le 3$.

(a) Write the first-degree Taylor polynomial for h about $x = 2$ and use it to approximate $h(1.9)$. Is this approximation greater than or less than $h(1.9)$? Explain your reasoning.

(b) Write the third-degree Taylor polynomial for h about $x = 2$ and use it to approximate $h(1.9)$.

(c) Use the Lagrange error bound to show that the third-degree Taylor polynomial for h about $x = 2$ approximates $h(1.9)$ with error less than 3×10^{-4}.

(a) $P_1(x) = 80 + 128(x - 2)$, so $h(1.9) \approx P_1(1.9) = 67.2$

$P_1(1.9) < h(1.9)$ since h' is increasing on the interval $1 \le x \le 3$.

$$4 : \begin{cases} 2 : P_1(x) \\ 1 : P_1(1.9) \\ 1 : P_1(1.9) < h(1.9) \text{ with reason} \end{cases}$$

(b) $P_3(x) = 80 + 128(x - 2) + \frac{488}{6}(x - 2)^2 + \frac{448}{18}(x - 2)^3$

$h(1.9) \approx P_3(1.9) = 67.988$

$$3 : \begin{cases} 2 : P_3(x) \\ 1 : P_3(1.9) \end{cases}$$

(c) The fourth derivative of h is increasing on the interval $1 \le x \le 3$, so $\max\limits_{1.9 \le x \le 2} \left| h^{(4)}(x) \right| = \frac{584}{9}$.

Therefore, $\left| h(1.9) - P_3(1.9) \right| \le \frac{584}{9} \frac{|1.9 - 2|^4}{4!}$

$$= 2.7037 \times 10^{-4}$$

$$< 3 \times 10^{-4}$$

$$2 : \begin{cases} 1 : \text{form of Lagrange error estimate} \\ 1 : \text{reasoning} \end{cases}$$