HW 6.10

P. 475 # 1-17 odd, 27

#1 $\int_0^\infty \frac{3x}{x^2+1}dx = \int_0^1 \frac{2x}{x^2+1}dx + \lim_{a\to\infty}\int_1^a \frac{2x}{x^2+1}dx$

oops! I thought we needed to write a sum of integrals

$u = x^2+1$ $\quad = \int_1^2 \frac{2x}{u}\frac{1}{2x}du + \lim_{a\to\infty}\int_2^{a^2+1}\frac{2x}{u}\frac{1}{2x}du$

$\frac{du}{dx} = 2x$ $\quad = \int_1^2 \frac{1}{u}du + \lim_{a\to\infty}\int_2^{a^2+1}\frac{1}{u}du = \lim_{a\to\infty}\left(\ln|u|\Big|_1^2 + \ln|u|\Big|_2^{a^2+1}\right)$

$\frac{1}{2x}du = dx$ $\quad = \lim_{a\to\infty}\left(\ln(2) - \ln(1) + \ln(a^2+1) - \ln(2)\right)$

$= \lim_{a\to\infty} \ln(a^2+1) \Rightarrow \boxed{\infty \text{ diverges}}$

Could have been $\lim_{a\to\infty}\int_1^a \frac{1}{u}du = \lim_{a\to\infty}\ln|u|\Big|_1^a = \lim_{a\to\infty}\ln(a) - \ln(1) \Rightarrow \boxed{\infty \text{ diverges}}$

where $u = x^2+1$

#3 $\int_{-\infty}^{\infty}\frac{2x}{(x^2+1)^2}dx = \lim_{a\to-\infty}\int_a^0 \frac{2x}{(x^2+1)^2}dx + \lim_{b\to\infty}\int_0^b \frac{2x}{(x^2+1)^2}dx$

$u = x^2+1$ $\quad = \lim_{a\to-\infty}\int_a^1 \frac{2x}{u^2}\frac{1}{2x}du + \lim_{b\to\infty}\int_1^b \frac{2x}{u^2}\frac{1}{2x}du$

$\frac{du}{dx} = 2x$ $\quad = \lim_{a\to-\infty}\int_a^1 \frac{1}{u^2}du + \lim_{b\to\infty}\int_1^b \frac{1}{u^2}du$

$\frac{1}{2x}du = dx$ $\quad = \lim_{a\to-\infty}\left(-u^{-1}\right)\Big|_a^1 + \lim_{b\to\infty}\left(-u^{-1}\right)\Big|_1^b$

$= \lim_{a\to-\infty}\left(-\frac{1}{1} - \left(-\frac{1}{a}\right)\right) + \lim_{b\to\infty}\left(-\frac{1}{b} - \left(-\frac{1}{1}\right)\right)$

$= -1 + 1 = \boxed{0 \text{ converges}}$

#5 $\int_1^\infty \frac{dx}{x^4} = \lim_{a\to\infty}\int_1^a \frac{1}{x^4}dx = \lim_{a\to\infty}\int_1^a x^{-4}dx = \lim_{a\to\infty}\left(-\frac{1}{3}x^{-3}\right)\Big|_1^a = \lim_{a\to\infty}\left(-\frac{1}{3a^3} - \left(-\frac{1}{3(1)^3}\right)\right)$

$= \frac{1}{3}$

#7 $\int_1^\infty \frac{dx}{\sqrt[3]{x}} = \lim_{a\to\infty}\int_1^a x^{-1/3}dx = \lim_{a\to\infty}\frac{3}{2}x^{2/3}\Big|_1^a = \lim_{a\to\infty}\left(\frac{3}{2}(a)^{2/3} - \frac{3}{2}(1)^{2/3}\right) \Rightarrow \boxed{\infty, \text{ diverges}}$

#9 $\int_{-\infty}^{-1}\frac{dx}{x^2} = \lim_{a\to-\infty}\int_a^{-1}x^{-2}dx = \lim_{a\to-\infty}\left(-x^{-1}\right)\Big|_a^{-1} = \lim_{a\to-\infty}\left(\frac{-1}{-1} - \left(\frac{-1}{a}\right)\right) = 1$

#11 $\int_{-\infty}^{-2}\frac{2}{x^2-1}dx = \lim_{a\to-\infty}\int_a^{-2}\frac{2}{x^2-1}dx = \lim_{a\to-\infty}\int_a^{-2}\frac{A}{x+1} + \frac{B}{x-1}dx$

$2 = (x-1)A + (x+1)B$ $\quad = \lim_{a\to-\infty}\int_a^{-2}\frac{-1}{x+1} + \frac{1}{x-1}dx$

$= \lim_{a\to-\infty}\left(-\ln|x+1| + \ln|x-1|\right)\Big|_a^{-2}$

$x=1 \quad B=1$

$x=-1 \quad A=-1$ $\quad = \lim_{a\to-\infty}\left(-\ln(1) + \ln(3) - \left(-\ln|a+1| + \ln|a-1|\right)\right)$

$= \lim_{a\to-\infty}\left(\ln(3) + \cancel{\ln|a+1| - \ln|a-1|}\right) = \ln(3)$

#13 $\int_{-1}^{\infty} \frac{dx}{x^2+5x+6} = \lim_{a\to\infty} \int_{-1}^{a} \frac{1}{x^2+5x+6} dx = \lim_{a\to\infty} \int_{-1}^{a} \frac{A}{x+2} + \frac{B}{x+3} dx = \lim_{a\to\infty} \int_{-1}^{a} \frac{1}{x+2} + \frac{-1}{x+3} dx$

$1 = A(x+3) + B(x+2)$ $\quad = \lim_{a\to\infty} \left[\ln|x+2| - \ln|x+3| \right] \Big|_{-1}^{a}$

$x=-3 \quad B=-1$ $\quad = \lim_{a\to\infty} \left[(\cancel{\ln|a+2|} - \cancel{\ln|a+3|}) - (\ln(1) - \ln(2)) \right]$

$x=-2 \quad A=1$ $\quad = \ln(2)$

#15 $\int_{1}^{\infty} \frac{5x+6}{x^2+2x} dx = \lim_{a\to\infty} \int_{1}^{a} \frac{5x+6}{x^2+2x} dx = \lim_{a\to\infty} \int_{1}^{a} \frac{A}{x} + \frac{B}{x+2} dx = \lim_{a\to\infty} \int_{1}^{a} \frac{3}{x} + \frac{2}{x+2} dx$

$5x+6 = A(x+2) + Bx$ $\quad = \lim_{a\to\infty} \left[3\ln|x| + 2\ln|x+2| \right] \Big|_{1}^{a}$

$x=0 \quad A=3$ $\quad = \lim_{a\to\infty} \left[(3\ln|a| + 2\ln|a+2|) - (3\ln|1| + 2\ln|3|) \right]$

$x=-2 \quad B=2$ $\quad \Rightarrow \boxed{\infty \text{ diverges}}$

#17 $\int_{1}^{\infty} x e^{-2x} dx = \lim_{a\to\infty} \int_{1}^{a} x e^{-2x} dx$ Int. by Parts $u = x$ $\quad dv = e^{-2x} dx$

$du = dx \quad v = -\frac{1}{2} e^{-2x}$

$= \lim_{a\to\infty} \left[\left(\frac{-x}{2} e^{-2x} \right) \Big|_{1}^{a} - \int_{1}^{a} -\frac{1}{2} e^{-2x} dx \right]$

$= \lim_{a\to\infty} \left[\left(\frac{-x}{2} e^{-2x} \right) \Big|_{1}^{a} + \frac{1}{2} \int_{1}^{a} e^{-2x} dx \right]$

$= \lim_{a\to\infty} \left[\left(\frac{-x}{2} e^{-2x} \right) \Big|_{1}^{a} + \frac{1}{2} \left(-\frac{1}{2} e^{-2x} \right) \Big|_{1}^{a} \right] = \lim_{a\to\infty} \left[\left(\frac{-a}{2} e^{-2a} \right) - \left(\frac{-1}{2} e^{-2} \right) + \frac{1}{2}\left(-\frac{1}{2} e^{-2a} \right) - \frac{1}{2}\left(-\frac{1}{2} e^{-2} \right) \right]$

$= \lim_{a\to\infty} \left[\underset{\downarrow \, 0}{\frac{-a}{2e^{2a}}} + \frac{1}{2e^2} - \underset{\downarrow \, 0}{\frac{1}{4e^{2a}}} + \frac{1}{4e^2} \right] = \frac{1}{2e^2} + \frac{1}{4e^2} = \frac{3}{4e^2}$

#27 $\int_{0}^{1} \frac{x+1}{\sqrt{x^2+2x}} dx$ improper b/c $x \neq 0$; infinite discont at end point $x=0$.

$\lim_{a\to 0^+} \int_{a}^{1} \frac{x+1}{\sqrt{x^2+2x}} dx = \lim_{a\to 0^+} \int_{a^2+2a}^{3} \frac{x+1}{\sqrt{u}} \frac{1}{2(x+1)} du = \lim_{a\to 0^+} \int_{a^2+2a}^{3} \frac{1}{2} u^{-1/2} du = \lim_{a\to 0^+} \left(u^{1/2} \right) \Big|_{a^2+2a}^{3}$

$u = x^2+2x$

$\frac{du}{dx} = 2x+2$ $\quad = \lim_{a\to 0^+} \left(\sqrt{3} - \sqrt{a^2+2a} \right) = \sqrt{3}$

$\frac{1}{2x+2} du = dx$

$\frac{1}{2(x+1)} du = dx$