## Homework

9. The functions $f$ and $g$ are differentiable. Given that $g(x) = f^{-1}(x)$, $f(1) = 3$, and $f'(1) = -5$, find $g'(3)$.

$g'(3) = \frac{1}{f'(f^{-1}(3))} = \frac{1}{f'(1)} = \frac{1}{-5}$

10. The functions $f$ and $g$ are differentiable. Given that $g(x) = f^{-1}(x)$, $f(2) = 4$, $f(4) = -6$, $f'(2) = 7$, and $f'(4) = 11$, find $g'(4)$.

$g'(4) = \frac{1}{f'(f^{-1}(4))} = \frac{1}{f'(2)} = \frac{1}{7}$

11. Selected values of a strictly monotonic function $h(x)$ and its derivative $h'(x)$ are shown on the table below.

| $x$ | $-1$ | $0$ | $2$ | $4$ |
|---|---|---|---|---|
| $h(x)$ | $-5$ | $-1$ | $4$ | $7$ |
| $h'(x)$ | $3$ | $\frac{1}{2}$ | $\frac{1}{6}$ | $5$ |

Let $f(x)$ be a function such that $f(x) = h^{-1}(x)$.
a) Find $f'(-1)$
b) Find $f'(4)$

a) $f'(-1) = \frac{1}{h'(h^{-1}(-1))} = \frac{1}{h'(0)} = \frac{1}{\frac{1}{2}} = \boxed{2}$

b) $f'(4) = \frac{1}{h'(h^{-1}(4))} = \frac{1}{h'(2)} = \frac{1}{\frac{1}{6}} = \boxed{6}$

1. If $f(x) = x^2$, find a derivative of $f^{-1}(x)$ at $x=16$

$y = x^2$
$x = y^2$
$\pm\sqrt{x} = y$ there are two branches → I'll use $y = \sqrt{x}$

$\frac{dy}{dx} = \frac{1}{2}x^{-1/2}$ at $x=16$ $\frac{dy}{dx} = \frac{1}{2} \cdot \frac{1}{\sqrt{16}} = \boxed{\frac{1}{8}}$

2. Suppose f(2)=7, f(1)=4, f'(2)=-4, f'(1)=1, determine $\frac{d}{dx} f^{-1}(7)$.

$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{f'(2)} = \boxed{-\frac{1}{4}}$

$x = 7$

3. The table below gives values of f(x), g(x), f'(x), and g'(x) at x=3.

| f(x) | g(x) | f'(x) | g'(x) |
|---|---|---|---|
| 1 | 4 | -2 | 5 |

a. Use the table to find v'(3) if $v(x) = \frac{f(x)}{g(x)}$.

$v'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2} \Rightarrow \frac{4(-2) - (1)(5)}{(4)^2} = \frac{-13}{16}$

b. Use the table to find h'(3) if $h(x) = 3xg(x)$.

$\underbrace{3xg(x)}_{\text{product}}$

$h'(x) = 3xg'(x) + 3g(x) \Rightarrow h'(3) = 3 \cdot 3g'(3) + 3g(3) = 9(5) + 3(4) = 57$

c. Use the table to find h'(3) if $h(x) = f(x)g(x) - 2x^2$.

$h'(x) = f(x)g'(x) + f'(x)g(x) - 4x \Rightarrow h'(3) = f(3)g'(3) + f'(3)g(3) - 4(3) = 1 \cdot 5 + (-2)4 - 12 = -15$

d. Use the table to find h'(3) if $h(x) = e^x f(x)$. Leave your answer in terms of e.

$h'(x) = e^x f'(x) + e^x f(x) \Rightarrow h'(3) = e^3 f'(3) + e^3 f(3) = e^3(-2) + e^3(1) = -e^3$

# 3.5 WS ans.

1 a. The chain rule is used to take derivatives of composite functions. To take the derivative of a composite function such as $f(g(x))$ with respect to $x$, find the derivative of $f(x)$ but evaluated at $g(x)$. Then multiply that value by $g'(x)$. This would look like $f'(g(x))g'(x)$.

b. i. $h(x) = \sqrt{(f(x))^2+7} = [(f(x))^2+7]^{1/2}$

$h'(x) = \frac{1}{2}((f(x))^2+7)^{-1/2} \cdot [2f(x) \cdot f'(x)]$

$h'(2) = \frac{1}{2}(3^2+7)^{-1/2} \cdot [2 \cdot 3 \cdot (-1)] = \frac{1}{2}(16)^{-1/2} \cdot (-6) = \frac{1}{2}\left(\frac{1}{4}\right) \cdot (-6) = \frac{-3}{4}$

ii. $l(x) = f(x^3 \cdot g(x))$ The derivative of the inside is a chain

$l'(x) = f'(x^3 g(x)) \cdot [x^3 g'(x) + 3x^2 g(x)]$

$l'(2) = f'(8 \cdot g(2)) \cdot [8g'(2) + 12g(2)] = f'(2) \cdot [16+3] = -19$

2 a) $f(g(x)) = \sec(x^3-2x+1)$

b) $f(x) = \sec(x)$

$f'(x) = \sec(x)\tan(x)$

c) $g(x) = x^3 - 2x+1$

$g'(x) = 3x^2-2$

d) $f(g(x)) = \sec(x^3-2x+1)$

$f'(g(x)) = \sec(x^3-2x+1)\tan(x^3-2x+1)$ Note: we are not asked for the derivative of the whole composite ∴ not chain rule

e) $(f \circ g)'(x) = \frac{d}{dx} f(g(x))$ Chain rule!

$f(g(x)) = \sec(x^3-2x+1)$

$f'(g(x)) \cdot g'(x) = \sec(x^3-2x+1)\tan(x^3-2x+1) \cdot [3x^2-2]$

3. a) $f(x) = \sqrt[3]{2x^3+7x+3} = (2x^3+7x+3)^{1/3}$

$f'(x) = \frac{1}{3}(2x^3+7x+3)^{-2/3} \cdot (6x^2+7) = \frac{6x^2+7}{3\sqrt[3]{(2x^3+7x+3)^2}}$

b) $g(t) = \tan(\sin(t))$

$g'(t) = \sec^2(\sin(t)) \cdot \cos(t)$

c) $h(u) = \sec^2(u) + \tan^2(u)$ ⟶ Chain Rules!

$h'(u) = 2\sec(u) \cdot \sec(u)\tan(u) + 2\tan(u) \cdot \sec^2(u) = 2\sec^2(u)\tan(u) + 2\sec^2(u)\tan(u)$

$h'(u) = 4\sec^2(u)\tan(u)$

3. d) $f(x) = xe^{3x^2+x}$

$f'(x) = x\, e^{3x^2+x} \cdot [6x+1] + 1 \cdot e^{3x^2+x} = e^{3x^2+x}[x[6x+1]+1] = e^{3x^2+x}[6x^2+x+1]$

e) $g(x) = \sin(\sin(\sin(x)))$

$g'(x) = \cos(\sin(\sin(x))) \cdot [\cos(\sin(x))] \cdot \cos(x)$

4. a) point: $f(2) = 2^2 \cdot e^6 = 4e^6$ $\quad (2, 4e^6)$

slope: $f'(x) = x^2 e^{3x} \cdot 3 + 2x e^{3x}$

$f'(2) = 4e^6 \cdot 3 + 4e^6 = 12e^6 + 4e^6 = 16e^6$

$T: y - 4e^6 = 16e^6(x-2)$

b) point: $f(0) = \sin(0) + \sin^2(0) = 0$ $\quad (0,0)$

slope: $f'(x) = \cos(x) + 2\sin(x)\cos(x)$

$f'(0) = \cos(0) + 2\sin(0)\cos(0) = 1 + 0 = 1$

$T: y - 0 = 1(x-0)$

5. a) $y = \dfrac{x}{x^2+1}$

$\dfrac{dy}{dx} = \dfrac{(x^2+1)(1) - x(2x)}{(x^2+1)^2} = \dfrac{x^2+1-2x^2}{(x^2+1)^2} = \dfrac{-x^2+1}{(x^2+1)^2}$

Same!

b) $y = x(x^2+1)^{-1}$

$\dfrac{dy}{dx} = x[-1\cdot(x^2+1)^{-2} \cdot 2x] + 1\cdot(x^2+1)^{-1} = \dfrac{-2x^2}{(x^2+1)^2} + \dfrac{1}{(x^2+1)} = \dfrac{-2x^2}{(x^2+1)^2} + \dfrac{x^2+1}{(x^2+1)^2} = \dfrac{-x^2+1}{(x^2+1)^2}$

6. $h(x) = \sqrt{4+3f(x)} = (4+3f(x))^{1/2}$

$h'(x) = \frac{1}{2}(4+3f(x))^{-1/2} \cdot [3f'(x)] \Rightarrow h'(1) = \frac{1}{2}[4+3\cdot 7]^{-1/2} \cdot [3\cdot 4] = \frac{1}{2} \cdot \frac{1}{\sqrt{25}} \cdot 12 = \frac{6}{5}$

7. $h(x) = f(g(x))$

$h'(x) = f'(g(x)) \cdot g'(x)$

$h'(-1) = f'(g(-1)) \cdot g'(-1) = f'(4) \cdot (-1) = -2$

$h'(3) = f'(g(3)) \cdot g'(3) = f'(-1)(-1) = 1$

$K(x) = g(f(x))$

$K'(x) = g'(f(x)) \cdot f'(x)$

$K'(2) = g'(f(2)) \cdot f'(2)$

$K'(2) = g'(3) \cdot (3) = (-1)(3) = -3$

8. $f(x) = 2\sin(x) + \sin^2(x)$ Horizontal tangent means $f'(x) = 0$

$f'(x) = 2\cos(x) + 2\sin(x)\cos(x)$

$f'(x) = 2\cos(x)[1+\sin(x)] = 0$

$2\cos(x) = 0$ $1+\sin(x) = 0$

$\cos(x) = 0$ $\sin(x) = -1$

$x = \frac{\pi}{2} + \pi n,\ n \in \mathbb{Z}$ $x = \frac{3\pi}{2} + 2\pi n,\ n \in \mathbb{Z}$

integers

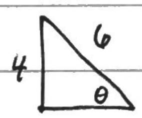
but this is covered by $\boxed{x = \frac{\pi}{2} + \pi n,\ n \in \mathbb{Z}}$

This means: $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$, etc.

9. a) Not necessarily, $f(\theta)$ could have been $f(\theta) = \cos(\theta) + 7$ or $f(\theta) = \cos(\theta) + \frac{1}{2}$

False

b)




False $\sin(\theta) = \frac{2}{3}$ simply indicates the ratio of opposite to hypoten.

$\sin(\theta) = \frac{2}{3}$ does not mean the opposite side $= 2$ and hypotenuse $= 3$.

c) $\tan(x) = \frac{\sin(x)}{\cos(x)}$

$$\sec^2(x) = \frac{\cos(x)\cos(x) - \sin(x)(-\sin(x))}{\cos^2(x)} = \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)}$$

so $\sec^2(x) = \frac{1}{\cos^2(x)}$ OR $\sec^2(x) = 1 + \frac{\sin^2(x)}{\cos^2(x)} = 1 + \tan^2(x)$