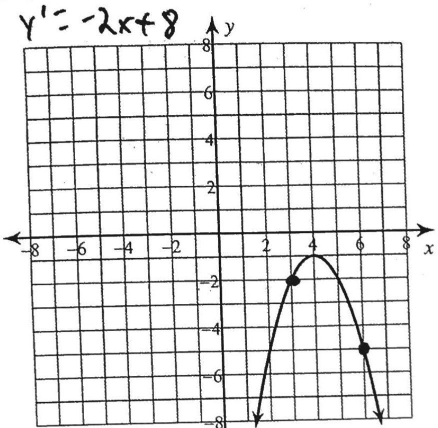
Mean Value Theorem

For each problem, find the values of c that satisfy the Mean Value Theorem.

1) $y = -x^2 + 8x - 17$; $[3, 6]$

$y' = -2x + 8$



$f'(c) = \dfrac{f(b) - f(a)}{b - a}$

$-2x + 8 = \dfrac{-5 + 2}{6 - 3}$

$-2x + 8 = \dfrac{-3}{3}$

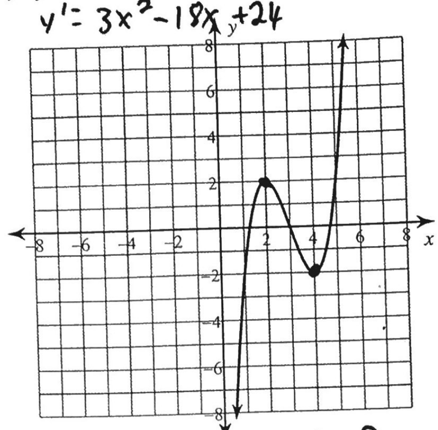
$-2x + 8 = -1$

$-2x = -9$

$c = \boxed{x = \frac{9}{2}}$

2) $y = x^3 - 9x^2 + 24x - 18$; $[2, 4]$

$y' = 3x^2 - 18x + 24$



$3x^2 - 18x + 24 = \dfrac{-2 - 2}{4 - 2}$

$3x^2 - 18x + 24 = \dfrac{-4}{2}$

$3x^2 - 18x + 24 = -2$

$3x^2 - 18x + 26 = 0$

Quadratic Formula

$c = \boxed{x = \dfrac{9 \pm \sqrt{3}}{3}}$

3) $y = -\dfrac{x^2}{2} + x - \dfrac{1}{2}$; $[-2, 1]$

when $x = -2$, $y = -\frac{9}{2}$

when $x = 1$, $y = 0$

$y' = -x + 1$

$-x + 1 = \dfrac{0 + \frac{9}{2}}{1 + 2}$

$-x + 1 = \frac{9}{6}$

$-x + 1 = \frac{3}{2}$

$-x = \frac{1}{2}$

$c = \boxed{x = -\frac{1}{2}}$

4) $y = \dfrac{x^2}{2} - 2x - 1$; $[-1, 1]$

$x = 0$

5) $y = x^3 + 3x^2 - 2$; $[-2, 0]$

$x = \dfrac{-3 \pm \sqrt{3}}{3}$

6) $y = -x^3 + 4x^2 - 3$; $[0, 4]$

$x = \dfrac{8}{3}$

7) $y = \dfrac{x^2 - 9}{3x}$; $[1, 4]$

$x = 2$

8) $y = \dfrac{x^2}{2x - 4}$; $[-4, 1]$

$x = 2 - \sqrt{6}$

9) $y = -(-2x+6)^{\frac{1}{2}}$; $[-2, 3]$

$x = \frac{7}{4}$

10) $y = -(-5x+25)^{\frac{1}{2}}$; $[3, 5]$

$y = -\sqrt{w}$

$y = 0$

Chain rule!

$y' = -\frac{1}{2}(-5x+25)^{-1/2}(-5)$

$y' = \frac{5}{2\sqrt{-5x+25}}$

$\frac{5}{2\sqrt{-5x+25}} = \frac{0+\sqrt{10}}{5-3}$

$\frac{5}{2\sqrt{-5x+25}} = \frac{\sqrt{10}}{2}$

$10 = 2\sqrt{10}\sqrt{-5x+25}$

$100 = 40(-5x+25)$

$100 = -200x + 1000$

$-900 = -200x$

$\boxed{x = \frac{9}{2}}$

For each problem, determine if the Mean Value Theorem can be applied. If it can, find all values of *c* that satisfy the theorem. If it cannot, explain why not. Remember, for MVT to apply, the function must be continuous on the closed interval and differentiable on the open interval.

11) $y = -\frac{x^2}{4x+8}$; $[-3, -1]$

MVT cannot be applied b/c $y = -\frac{x^2}{4x+8}$ is not continuous on $[-3, -1]$. It is not continuous at $x = -2$.

12) $y = \frac{-x^2+9}{4x}$; $[1, 3]$

$y = \frac{-x}{4} + \frac{9}{4x}$

$y = \frac{-x}{4} + \frac{9}{4}x^{-1}$

$y' = -\frac{1}{4} - \frac{9}{4}x^{-2}$

The function is continuous and differentiable on $[1, 3]$

$-\frac{1}{4} - \frac{9}{4x^2} = \frac{0-2}{3-1}$

$-\frac{1}{4} - \frac{9}{4x^2} = -1$

$-\frac{9}{4x^2} = \frac{-3}{4}$

$-36 = -12x^2$

$3 = x^2$

$\boxed{x = \sqrt{3}}$ or $-\sqrt{3}$

not on domain

13) $y = -(6x+24)^{\frac{2}{3}}$; $[-4, -1]$

MVT applies

$x = \frac{-28}{9}$

14) $y = (x-3)^{\frac{2}{3}}$; $[1, 4]$

$y' = \frac{2}{3}(x-3)^{-1/3}$

$y' = \frac{2}{3\sqrt[3]{x-3}}$

The function is not differentiable on $(1, 4)$. At $x = 3$, the function is not differentiable.

Optimization

Name________________

1) Find two real numbers whose sum is 30 and whose product is maximized.

$x+y=30 \quad xy=P$

Max Product

$y=30-x$

$x(30-x)=P$

$30x-x^2=P$

$30-2x=P'$

$30-2x=0$

$30=2x$

$\boxed{x=15 \quad y=15}$

2) Find two numbers whose difference is 50 and whose product is minimized.

$x-y=50 \quad xy=P$

Min Product

$x=50+y$

$(50+y)y=P$

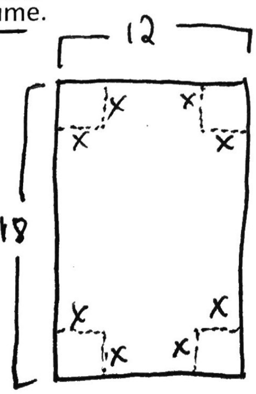
$50y+y^2=P$

$50+2y=P'$

$50+2y=0$

$\boxed{y=-25 \quad x=25}$

3) An open box with a rectangular base is to be constructed from a 12" by 18" piece of cardboard by cutting out squares from each corner and bending up the sides. Find the dimensions of the box that will have the largest volume.



$12-2x=\text{Length}$

$18-2x=\text{width}$

$x=\text{height}$

Max Volume

$(12-2x)(18-2x)x=V$

$(12-2x)(18x-2x^2)=V$

$216x-24x^2-36x^2+4x^3=V$

$4x^3-60x^2+216x=V$

$12x^2-120x+216=V'$

$12(x^2-10x+18)=V'$

$12(x^2-10x+18)=0$

Must use quadratic formula

$x=\dfrac{10\pm\sqrt{100-4(18)}}{2}$

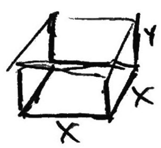
$x=\dfrac{10\pm\sqrt{28}}{2}$

$x=2.354$ or $x=7.646$ ↳ too big for box!

$\boxed{\begin{array}{l} x=2.354\text{ in} \\ \text{Length}=7.292\text{ in} \\ \text{width}=13.292\text{ in} \end{array}}$

this my height the distance around an object.

4) A rectangular package can be sent through the mail only if the sum of its length and girth is not more than 120". Find the dimensions of the box of maximum volume that can be sent if the base of the box is square.



$x^2 y = V$ $\quad 4x + y = 120$

$y = 120 - 4x$

$x^2(120-4x) = V$ $\quad$ Max Volume.

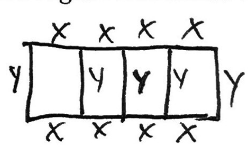
$120x^2 - 4x^3 = V$

$240x - 12x^2 = v'$

$12x(20 - x) = v'$

$x = 0$ or 20 $\quad$ if $x = 20,\ y = 40$ $\quad$ $20'' \times 20'' \times 40''$

5) A gardener wants to construct 4 garden areas by first building a fence around a rectangular region, then subdividing the region into 4 smaller rectangles by placing 3 fences parallel to one side of the rectangle. What dimensions of the region minimizes the amount of fencing if the total area of the region is 300 square feet?



$4xy = A$

$4xy = 300$

$y = \frac{75}{x}$

Min Fencing

$8x + 5y = F$

$8x + 5\left(\frac{75}{x}\right) = F$

$8x + 375x^{-1} = F$

$8 - 375x^{-2} = F'$

$8 - \frac{375}{x^2} = 0$

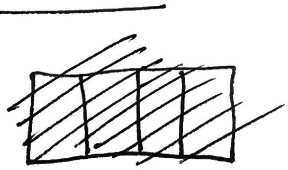
$8 = \frac{375}{x^2} \rightarrow x^2 = \frac{375}{8} \rightarrow x = \sqrt{\frac{375}{8}}$

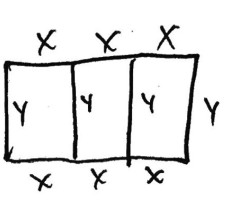
$4x = 27.386$ ft

$y = 10.954$ ft

$\boxed{27.386\text{ft} \times 10.954\text{ft}}$

6) A farmer has 150 feet of fencing and wants to construct 3 pig pens by first building a fence around a rectangular region, then subdividing the region into 3 smaller rectangles by placing 2 fences parallel to one side of the rectangle. What dimensions of the region maximizes the total area?





$6x + 4y = 150$

$3xy = A$

Max area

$4y = 150 - 6x$

$y = \frac{75}{2} - \frac{3}{2}x$

$3x\left(\frac{75}{2} - \frac{3}{2}x\right) = A$

$\frac{225x}{2} - \frac{9x^2}{2} = A$

$\frac{225}{2} - 9x = A'$

$\frac{225}{2} = 9x$

$\frac{225}{18} = x$

$12.5 = x$

$3x = 37.5$ft

$y = 18.75$ft

$\boxed{37.5\text{ft} \times 18.75\text{ft}}$