Remember: $\sum_{K=0}^{\infty} \frac{f^{(K)}(a)}{K!}(x-a)^K$

Pg. 500 # #1-4, 5, 13, 19 HW 10.7

#1

$f(0) = \sqrt{1} = 1$

$f'(0) = 0$

$f''(0) = 1$

$f'''(0) = 0$

$f^{(4)}(0) = -3$

$f'(x) = \frac{1}{2}(1+x^2)^{-1/2} \cdot 2x = \frac{x}{\sqrt{1+x^2}}$

$f''(x) = \frac{\sqrt{1+x^2} \cdot 1 - x\left(\frac{1}{2}(1+x^2)^{-1/2} \cdot 2x\right)}{1+x^2} = \frac{\sqrt{1+x^2} - \frac{x^2}{\sqrt{1+x^2}}}{1+x^2}$

$f'''(x) = \frac{(1+x^2)\left[\frac{1}{2}(x^2+1)^{-1/2} \cdot 2x - \left[\frac{\sqrt{1+x^2} \cdot 2x - x^2 \frac{1}{2}(1+x^2)^{-1/2} \cdot 2x}{1+x^2}\right]\right] - \left[\sqrt{1+x^2} - \frac{x^2}{\sqrt{1+x^2}}\right]2x}{[1+x^2]^2}$

These derivatives get bad but notice the derivative pattern! ↳ repeated quotient and chain rule

$f^{(4)}(0) = -3$; It is!

I'll let you work it out! Or use a calculator!

4th order $P_4(x) = \frac{1}{0!}x^0 + \frac{0}{1!}x^1 + \frac{1}{2!}x^2 + \frac{0}{3!}x^3 + \frac{-3}{4!}x^4 = 1 + \frac{1}{2}x^2 + \frac{-3}{24}x^4 = \boxed{1 + \frac{1}{2}x^2 - \frac{1}{8}x^4}$

#2.

$f(0) = e^0 = 1$

$f'(0) = 2e^0 = 2$

$f''(0) = 4$

$f'''(0) = 8$

$f^{(4)}(0) = 16$

$f'(x) = 2e^{2x}$

$f''(x) = 4e^{2x}$

$f'''(x) = 8e^{2x}$

$f^{(4)} = 16e^{2x}$

4th order $P_4(x) = \frac{1}{0!}x^0 + \frac{2}{1!}x^1 + \frac{4}{2!}x^2 + \frac{8}{3!}x^3 + \frac{16}{4!}x^4 = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4$

#3.

$f(0) = \frac{1}{2}$

$f'(0) = -\frac{1}{4}$

$f''(0) = \frac{1}{4}$

$f'''(0) = -\frac{3}{8}$

$f^{(4)}(0) = \frac{3}{4}$

$f^{(5)}(0) = \frac{-120}{64} = -\frac{60}{32} = \frac{-30}{16} = \frac{-15}{8}$

$f'(x) = -(x+2)^{-2}$

$f''(x) = 2(x+2)^{-3}$

$f'''(x) = -6(x+2)^{-4}$

$f^{(4)}(x) = 24(x+2)^{-5}$

$f^{(5)}(x) = -120(x+2)^{-6}$

5th order $P_5(x) = \frac{1/2}{0!}x^0 + \frac{-\frac{1}{4}}{1!}x^1 + \frac{\frac{1}{4}}{2!}x^2 + \frac{-\frac{3}{8}}{3!}x^3 + \frac{\frac{3}{4}}{4!}x^4 + \frac{-\frac{15}{8}}{5!}x^5$

$= \frac{1}{2} - \frac{1}{4}x + \frac{1}{8}x^2 - \frac{1}{16}x^3 + \frac{1}{32}x^4 - \frac{1}{64}x^5$

#4. $f(0) = e^1 = e$ $\quad f'(x) = -e^{1-x}$

$f'(0) = -e$ $\quad f''(x) = e^{1-x}$

$f''(0) = e$ $\quad f'''(x) = -e^{1-x}$

$f'''(0) = -e$ $\quad f^{(4)}(x) = e^{1-x}$

$f^{(4)}(0) = e$ $\quad f^{(5)}(x) = -e^{1-x}$

$f^{(5)}(0) = -e$

5th order $P_5(x) = \frac{e}{0!}x^0 + \frac{-e}{1!}x^1 + \frac{e}{2!}x^2 + \frac{-e}{3!}x^3 + \frac{e}{4!}x^4 + \frac{-e}{5!}x^5$

$$= e - ex + \frac{e}{2}x^2 - \frac{e}{6}x^3 + \frac{e}{24}x^4 - \frac{e}{120}x^5$$

#5. $\sin(2x)$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + (-1)^n\left(\frac{x^{2n+1}}{(2n+1)!}\right)$$

So... $\sin(2x) = 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \ldots + (-1)^n\left[\frac{(2x)^{2n+1}}{(2n+1)!}\right] + \ldots$

$= 2x - \frac{4x^3}{3} + \frac{4x^5}{15} - \ldots + (-1)^n\left[\frac{(2x)^{2n+1}}{(2n+1)!}\right] + \ldots$ converges for all real x ($\sin(x)$ converges that way too!)

#13. $f(x) = \frac{1}{x+1}$ at $x=2$

$f(2) = \frac{1}{3}$ $\quad f'(x) = -(x+1)^{-2}$

$f'(2) = -\frac{1}{9}$ $\quad f''(x) = 2(x+1)^{-3}$

$f''(2) = \frac{2}{27}$ $\quad f'''(x) = -6(x+1)^{-4}$

$f'''(2) = \frac{-6}{81} = \frac{-2}{27}$

$$P_n(x) = \frac{\frac{1}{3}}{0!}(x-2)^0 + \frac{-\frac{1}{9}}{1!}(x-2)^1 + \frac{\frac{2}{27}}{2!}(x-2)^2 + \frac{\frac{-2}{27}}{3!}(x-2)^3 + \ldots$$

$$= \frac{1}{3} - \frac{1}{9}(x-2) + \frac{1}{27}(x-2)^2 - \frac{1}{81}(x-2)^3 + \ldots \quad (-1)^n\frac{(x-2)^n}{3^{n+1}} + \ldots$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{(x-2)^n}{3^{n+1}}$$

#19 $f(\pi/4) = \frac{\sqrt{2}}{2}$ $f'(x) = \cos(x)$ $f(x) = \sin(x)$ at $x = \pi/4$

$f'(\pi/4) = \sqrt{2}/2$ $f''(x) = -\sin(x)$

$f''(\pi/4) = -\sqrt{2}/2$ $f'''(x) = -\cos(x)$

$f'''(\pi/4) = -\sqrt{2}/2$

$$P_0(x) = \frac{\sqrt{2}}{2}$$

$$P_1(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(x - \pi/4)$$

$$P_2(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(x - \pi/4) - \frac{\frac{\sqrt{2}}{2}}{2!}(x - \pi/4)^2 = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(x - \pi/4) - \frac{\sqrt{2}}{4}(x - \pi/4)^2$$

$$P_3(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(x - \pi/4) - \frac{\sqrt{2}}{4}(x - \pi/4)^2 - \frac{\frac{\sqrt{2}}{2}}{3!}(x - \pi/4)^3$$

$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(x - \pi/4) - \frac{\sqrt{2}}{4}(x - \pi/4)^2 - \frac{\sqrt{2}}{12}(x - \pi/4)^3$$