Question 4

t (seconds)	0	10	20	30	40	50	60	70	80
$v(t)$ (feet per second)	5	14	22	29	35	40	44	47	49

Rocket A has positive velocity $v(t)$ after being launched upward from an initial height of 0 feet at time $t = 0$ seconds. The velocity of the rocket is recorded for selected values of t over the interval $0 \le t \le 80$ seconds, as shown in the table above.

(a) Find the average acceleration of rocket A over the time interval $0 \le t \le 80$ seconds. Indicate units of measure.

(b) Using correct units, explain the meaning of $\int_{10}^{70} v(t)\,dt$ in terms of the rocket's flight. Use a midpoint Riemann sum with 3 subintervals of equal length to approximate $\int_{10}^{70} v(t)\,dt$.

(c) Rocket B is launched upward with an acceleration of $a(t) = \dfrac{3}{\sqrt{t+1}}$ feet per second per second. At time $t = 0$ seconds, the initial height of the rocket is 0 feet, and the initial velocity is 2 feet per second. Which of the two rockets is traveling faster at time $t = 80$ seconds? Explain your answer.

(a) Average acceleration of rocket A is

$$\frac{v(80) - v(0)}{80 - 0} = \frac{49 - 5}{80} = \frac{11}{20}\ \text{ft}/\text{sec}^2$$

1 : answer

(b) Since the velocity is positive, $\int_{10}^{70} v(t)\,dt$ represents the distance, in feet, traveled by rocket A from $t = 10$ seconds to $t = 70$ seconds.

A midpoint Riemann sum is

$$20[v(20) + v(40) + v(60)]$$
$$= 20[22 + 35 + 44] = 2020 \text{ ft}$$

$$3 : \begin{cases} 1 : \text{explanation} \\ 1 : \text{uses } v(20),\ v(40),\ v(60) \\ 1 : \text{value} \end{cases}$$

(c) Let $v_B(t)$ be the velocity of rocket B at time t.

$$v_B(t) = \int \frac{3}{\sqrt{t+1}}\,dt = 6\sqrt{t+1} + C$$
$$2 = v_B(0) = 6 + C$$
$$v_B(t) = 6\sqrt{t+1} - 4$$
$$v_B(80) = 50 > 49 = v(80)$$

Rocket B is traveling faster at time $t = 80$ seconds.

$$4 : \begin{cases} 1 : 6\sqrt{t+1} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{finds } v_B(80),\text{ compares to } v(80), \\ \quad\ \text{and draws a conclusion} \end{cases}$$

Units of ft/sec^2 in (a) and ft in (b)

1 : units in (a) and (b)

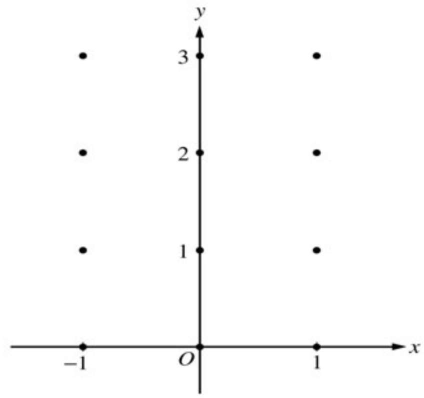
Question 6

Consider the differential equation $\frac{dy}{dx} = x^2(y-1)$.

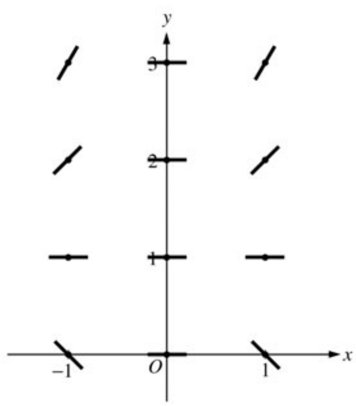
(a) On the axes provided, sketch a slope field for the given differential equation at the twelve points indicated.
(**Note: Use the axes provided in the pink test booklet.**)

(b) While the slope field in part (a) is drawn at only twelve points, it is defined at every point in the xy-plane. Describe all points in the xy-plane for which the slopes are positive.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(0) = 3$.



(a)



2 :
- 1 : zero slope at each point (x, y) where $x = 0$ or $y = 1$
- 1 :
  - positive slope at each point (x, y) where $x \neq 0$ and $y > 1$
  - negative slope at each point (x, y) where $x \neq 0$ and $y < 1$

(b) Slopes are positive at points (x, y) where $x \neq 0$ and $y > 1$.

1 : description

(c) $\frac{1}{y-1}dy = x^2 dx$

$\ln|y-1| = \frac{1}{3}x^3 + C$

$|y-1| = e^C e^{\frac{1}{3}x^3}$

$y - 1 = Ke^{\frac{1}{3}x^3}$, $K = \pm e^C$

$2 = Ke^0 = K$

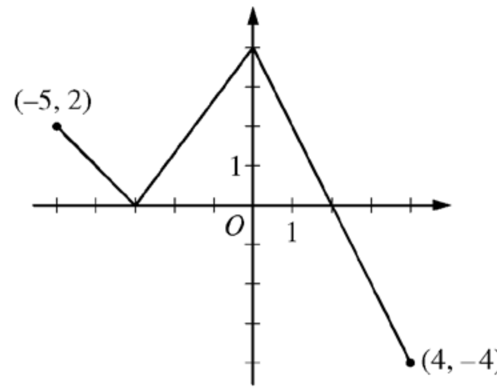
$y = 1 + 2e^{\frac{1}{3}x^3}$

6 :
- 1 : separates variables
- 2 : antiderivatives
- 1 : constant of integration
- 1 : uses initial condition
- 1 : solves for y
  - $0/1$ if y is not exponential

Note: max $3/6$ [1-2-0-0-0] if no constant of integration
Note: $0/6$ if no separation of variables

Question 3

The function f is defined on the closed interval $[-5, 4]$. The graph of f consists of three line segments and is shown in the figure above.



Graph of f

Let g be the function defined by $g(x) = \int_{-3}^{x} f(t)\, dt$.

(a) Find $g(3)$.

(b) On what open intervals contained in $-5 < x < 4$ is the graph of g both increasing and concave down? Give a reason for your answer.

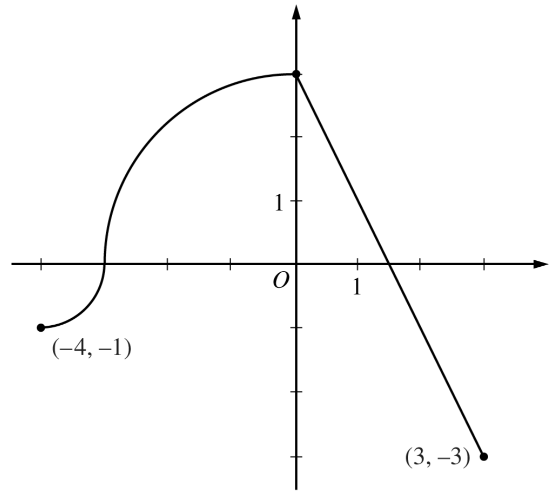
(c) The function h is defined by $h(x) = \dfrac{g(x)}{5x}$. Find $h'(3)$.

(d) The function p is defined by $p(x) = f\left(x^2 - x\right)$. Find the slope of the line tangent to the graph of p at the point where $x = -1$.

(a) $g(3) = \int_{-3}^{3} f(t)\, dt = 6 + 4 - 1 = 9$	1 : answer
(b) $g'(x) = f(x)$ The graph of g is increasing and concave down on the intervals $-5 < x < -3$ and $0 < x < 2$ because $g' = f$ is positive and decreasing on these intervals.	$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{reason} \end{cases}$
(c) $h'(x) = \dfrac{5xg'(x) - g(x)5}{(5x)^2} = \dfrac{5xg'(x) - 5g(x)}{25x^2}$ $h'(3) = \dfrac{(5)(3)g'(3) - 5g(3)}{25 \cdot 3^2}$ $= \dfrac{15(-2) - 5(9)}{225} = \dfrac{-75}{225} = -\dfrac{1}{3}$	$3 : \begin{cases} 2 : h'(x) \\ 1 : \text{answer} \end{cases}$
(d) $p'(x) = f'\left(x^2 - x\right)(2x - 1)$ $p'(-1) = f'(2)(-3) = (-2)(-3) = 6$	$3 : \begin{cases} 2 : p'(x) \\ 1 : \text{answer} \end{cases}$

Question 4

The continuous function f is defined on the interval $-4 \le x \le 3$. The graph of f consists of two quarter circles and one line segment, as shown in the figure above.



Graph of f

Let $g(x) = 2x + \int_0^x f(t)\,dt$.

(a) Find $g(-3)$. Find $g'(x)$ and evaluate $g'(-3)$.

(b) Determine the x-coordinate of the point at which g has an absolute maximum on the interval $-4 \le x \le 3$. Justify your answer.

(c) Find all values of x on the interval $-4 < x < 3$ for which the graph of g has a point of inflection. Give a reason for your answer.

(d) Find the average rate of change of f on the interval $-4 \le x \le 3$. There is no point c, $-4 < c < 3$, for which $f'(c)$ is equal to that average rate of change. Explain why this statement does not contradict the Mean Value Theorem.

(a) $g(-3) = 2(-3) + \int_0^{-3} f(t)\,dt = -6 - \frac{9\pi}{4}$ $g'(x) = 2 + f(x)$ $g'(-3) = 2 + f(-3) = 2$	$3: \begin{cases} 1: g(-3) \\ 1: g'(x) \\ 1: g'(-3) \end{cases}$
(b) $g'(x) = 0$ when $f(x) = -2$. This occurs at $x = \frac{5}{2}$. $g'(x) > 0$ for $-4 < x < \frac{5}{2}$ and $g'(x) < 0$ for $\frac{5}{2} < x < 3$. Therefore g has an absolute maximum at $x = \frac{5}{2}$.	$3: \begin{cases} 1: \text{considers } g'(x) = 0 \\ 1: \text{identifies interior candidate} \\ 1: \text{answer with justification} \end{cases}$
(c) $g''(x) = f'(x)$ changes sign only at $x = 0$. Thus the graph of g has a point of inflection at $x = 0$.	1 : answer with reason
(d) The average rate of change of f on the interval $-4 \le x \le 3$ is $\frac{f(3) - f(-4)}{3 - (-4)} = -\frac{2}{7}$. To apply the Mean Value Theorem, f must be differentiable at each point in the interval $-4 < x < 3$. However, f is not differentiable at $x = -3$ and $x = 0$.	$2: \begin{cases} 1: \text{average rate of change} \\ 1: \text{explanation} \end{cases}$