AP Exam Practice

AP Practice Problem

Consider the geometric series $\sum_{n=0}^{\infty} k\left(\frac{k+3}{6}\right)^n$ where k is a constant.

a.) Find $\sum_{n=0}^{\infty} k\left(\frac{k+3}{6}\right)^n$ when $k = 1$.

$$k=1 \Rightarrow \sum_{n=0}^{\infty} k\left(\frac{k+3}{6}\right)^n = \sum_{n=0}^{\infty} \left(\frac{4}{6}\right)^n = \frac{1}{1-\frac{4}{6}} = \frac{6}{6-4} = 3$$

b.) Find k when $\sum_{n=0}^{\infty} k\left(\frac{k+3}{6}\right)^n = 12$.

$$\sum_{n=0}^{\infty} k\left(\frac{k+3}{6}\right)^n = 12 \Rightarrow \text{ converges} \Rightarrow \left|\frac{k+3}{6}\right| < 1 \Rightarrow |k+3| < 6 \Rightarrow -9 < k < 3$$

$$\sum_{n=0}^{\infty} k\left(\frac{k+3}{6}\right)^n = 12 \Rightarrow \frac{k}{1-\frac{k+3}{6}} = 12 \Rightarrow \frac{6k}{6-(k+3)} = 12 \Rightarrow \frac{6k}{3-k} = 12 \Rightarrow 6k = 36-12k \Rightarrow k = 2$$

c.) The series $\sum_{n=0}^{\infty} k\left(\frac{k+3}{6}\right)^n$ converges for $a < k < b$ and diverges when $k = a$ or $k = b$. Find a and b.

$$\sum_{n=0}^{\infty} k\left(\frac{k+3}{6}\right)^n \text{ converges} \Rightarrow \left|\frac{k+3}{6}\right| < 1 \Rightarrow |k+3| < 6 \Rightarrow -9 < k < 3 \Rightarrow a = -9, b = 3$$

Practice 3: Each statement below if false. Correct each statement to create a true statement.

For Statements 1 – 3: Let $a_n > 0$

Statement 1: If $a_{n+1} \leq a_n$ and $\lim_{n\to\infty} a_n$ converges, then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges

If $a_{n+1} \leq a_n$ and $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges

Statement 2: If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges

If $a_{n+1} \leq a_n$ and $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges

Statement 3: If $\sum_{n=1}^{\infty} (-1)^n a_n$ diverges, then $\lim_{n\to\infty} a_n = 0$

$\sum_{n=1}^{\infty} (-1)^n a_n$ converges, if $\lim_{n\to\infty} a_n = 0$

Statement 4: Consider the series $\sum_{n=1}^{\infty} b_n$. If $\sum_{n=1}^{\infty} b_n$ diverges, then $\lim_{n\to\infty} b_n \neq 0$

$\sum_{n=1}^{\infty} b_n$ diverges, if $\lim_{n\to\infty} b_n \neq 0$

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Consider the altnerating series defined below:

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

A) Use the alternating series test to show that this series converges when $x = 3$.

$$\sum_{n=0}^{\infty} \frac{(-1)^n (3)^{2n}}{(2n)!} = 1 - \frac{(3)^2}{2} + \frac{(3)^4}{4!} - \frac{(3)^6}{6!} + \frac{(3)^8}{8!} - \cdots 1 - \frac{9}{2} + \frac{81}{24} - \frac{729}{720} + \frac{6561}{40320} - \cdots$$

$a_{n+1} < a_n$ when $n > 1$ $\qquad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3^{2n}}{(2n)!} = 0 \Rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n (3)^{2n}}{(2n)!}$ converges

B) Show that this series converges for all x values where x is a real number.

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

$a_{n+1} \le a_n$ when $n > N$ where N is an integer.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{x^{2n}}{(2n)!} = 0$$

$$\frac{x^{2(n+1)}}{(2(n+1))!} < \frac{x^{2n}}{(2n)!} \Rightarrow \frac{(2n)!}{(2n+2)!} < \frac{x^{2n}}{x^{2n+2}} \Rightarrow \frac{x^{2n+2}}{x^{2n}} < \frac{(2n+2)!}{(2n)!} \Rightarrow x^2 < (2n+2)(2n+1)$$

$$0 < (2n+2)(2n+1) - x^2 \Rightarrow 4n^2 + 6n + (2 - x^2) > 0$$

$4n^2 + 6n + (2 - x^2)$ is an open up parabola so when $n >$ the positive zero to the right

then $4n^2 + 6n + (2 - x^2) > 0. \Rightarrow$

$$N = \frac{-6 + \sqrt{6^2 - 4(4)(2 - x^2)}}{2(4)} = \frac{-6 + \sqrt{36 - 32 + 16x^2}}{2(4)} = \frac{-6 + \sqrt{4 + 16x^2}}{2(4)} = \frac{-3 + \sqrt{4x^2 + 1}}{4}$$

N can be found for any value of x.

C) Consider the function $f(x)$ where $f'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

Determine if $f(x)$ has a relative minimum, relative maximum or neither at $x = 0$.
Give a reason for your answer.

$$f'(0) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{(0)^2}{2} + \frac{(0)^4}{4!} - \frac{(0)^6}{6!} + \cdots = 1 \Rightarrow \text{Neither}$$

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$$\text{Let } a(n) = \frac{1}{n^{k+1}} \text{ where } k \text{ is a constant}$$

(a) For $k = \frac{1}{2}$, use the alternating series test to show that $\sum_{n=1}^{\infty}(-1)^n a(n)$ converges. Determine if this series converges conditionally or converges absolutely. Explain your reasoning.

$$k = \frac{1}{2} \Rightarrow \sum_{n=1}^{\infty}(-1)^n a(n) = \sum_{n=1}^{\infty}(-1)^n \frac{1}{n^{3/2}}$$

$$\lim_{n\to\infty}\frac{1}{n^{3/2}} \to \frac{1}{\infty} \Rightarrow \lim_{n\to\infty}\frac{1}{n^{3/2}} = 0 \qquad \frac{1}{(n+1)^{3/2}} \le \frac{1}{n^{3/2}} \Rightarrow a(n+1) \le a(n)$$

$\sum_{n=1}^{\infty}(-1)^n \frac{1}{n^{3/2}}$ is a convergent alternating series

$\sum_{n=1}^{\infty}\left|(-1)^n \frac{1}{n^{3/2}}\right| = \sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ convergent $p-$series with $p = \frac{3}{2} > 1$

$\sum_{n=1}^{\infty}(-1)^n \frac{1}{n^{3/2}}$ converges absolutely

(b) Let $b(n) = a(\sqrt{n})$. Find all integer values of k such that $\sum_{n=1}^{\infty}(-1)^n b(n)$ converges conditionally.

$$\sum_{n=1}^{\infty}(-1)^n b(n) = \sum_{n=1}^{\infty}(-1)^n \frac{1}{\left(\sqrt{n}\right)^{k+1}} = \sum_{n=1}^{\infty}(-1)^n \frac{1}{n^{\frac{1}{2}(k+1)}}$$

converges conditionally $\Rightarrow \sum_{n=1}^{\infty}(-1)^n \frac{1}{n^{\frac{1}{2}(k+1)}}$ converges but $\sum_{n=1}^{\infty}\frac{1}{n^{\frac{1}{2}(k+1)}}$ does not converge.

$$\lim_{n\to\infty}\frac{1}{n^{\frac{1}{2}(k+1)}} = 0 \Rightarrow \lim_{n\to\infty} n^{\frac{1}{2}(k+1)} \to \infty \Rightarrow \frac{1}{2}(k+1) > 0 \Rightarrow k > -1 \qquad \frac{1}{(n+1)^{\frac{1}{2}(k+1)}} \le \frac{1}{n^{\frac{1}{2}(k+1)}} \Rightarrow b(n+1) \le b(n)$$

$\sum_{n=1}^{\infty}\frac{1}{n^{\frac{1}{2}(k+1)}}$ is a divergent $p-$series with $p = \frac{1}{2}(k+1) \le 1 \Rightarrow k \le 1$

$-1 < k \le 1 \Rightarrow k = 0,1$

(c) Let $c(n) = a(n^{-2k})$. Show that there is no real value of k such that $\sum_{n=1}^{\infty} c(n)$ is the harmonic series.

$$\sum_{n=1}^{\infty} c(n) = \sum_{n=1}^{\infty}\frac{1}{\left(n^{-2k}\right)^{k+1}} = \sum_{n=1}^{\infty}\frac{1}{n^{-2k(k+1)}}$$

Harmonic series $\Rightarrow -2k(k+1) = 1$

$0 = 2k^2 + 2k + 1 \Rightarrow k = \frac{-2 \pm \sqrt{2^2 - 4(2)(1)}}{2(2)} = \frac{-2 \pm \sqrt{-4}}{4}$ which is not a real number