AP Review 13: Series and Convergence Tests

Today we will review Geometric Series, Harmonic and *p*-Series, and the Alternating Series Test.

CONVERGENCE OF A GEOMETRIC SERIES

1. If $|r| < 1$, the geometric series $\sum_{n=0}^{\infty} ar^n$ converges

2. If $|r| \geq 1$, the geometric series $\sum_{n=0}^{\infty} ar^n$ diverges.

Example 1: For each of the following geometric series, find the value of *r* and determine if the series converges or diverges.

a.) $\sum_{n=0}^{\infty} 4\left(\frac{2}{3}\right)^{n-1}$	**b.)** $\sum_{n=0}^{\infty} \frac{3(2)^n}{e^{n+1}}$	**c.)** $\sum_{n=0}^{\infty} \frac{(-1)^n}{3}$	**d.)** $\sum_{n=0}^{\infty} \frac{-5}{3^n}$

e.) $\sum_{n=0}^{\infty} \frac{(-1)^n}{6^{n-1}}$	**f.)** $\sum_{n=0}^{\infty} \frac{\pi^{n-1}}{e^n}$

SUM OF AN INFINITE GEOMETRIC SERIES

If $|r| < 1$, the geometric series $\sum_{n=0}^{\infty} ar^n$ converges, and its sum is

$$S = \frac{a}{1-r},$$

where *a* is the first term of the geometric series and *r* is the common ratio.

Example 2: Determine if the following geometric series converge or diverge. If the series converges, find the sum.

a.) $\sum_{n=0}^{\infty} \frac{2}{4^n}$	**b.)** $\sum_{n=2}^{\infty} \frac{\pi^n}{e^{2n-4}}$	**c.)** $\sum_{n=0}^{\infty} \frac{2(3)^n}{e^n}$	**d.)** $\sum_{n=1}^{\infty} e\left(\frac{-1}{2}\right)^{n-1}$

Practice 1: Consider the geometric series $\sum_{n=0}^{\infty} 2\left(\frac{k}{3}\right)^n$ where k is a constant.

a.) Find k such that $\sum_{n=0}^{\infty} 2\left(\frac{k}{3}\right)^n = 3$.	**b.)** Find k such that $\sum_{n=0}^{\infty} 2\left(\frac{k}{3}\right)^n = \frac{4}{3}$.

CONVERGENCE OF A *p*-SERIES

The p-series is defined by the following where p is a positive real number.

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

1. converges if $p > 1$, and

2. diverges if $0 < p \leq 1$.

Example 1: Determine if the following series converge or diverge. Identify any value(s) for p.

a.) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$	**b.)** $\sum_{n=1}^{\infty} \frac{-2}{n^3}$	**c.)** $\sum_{n=1}^{\infty} n^{-2} \cdot \sqrt{n}$

Alternating Series Test

Let $a_n > 0$. The alternating series

$$\sum_{n=1}^{\infty}(-1)^n a_n \quad \text{and} \quad \sum_{n=1}^{\infty}(-1)^{n-1} a_n$$

converge if the following conditions are both met:

1. $\lim_{n\to\infty} a_n = 0$ 2. $a_{n+1} \le a_n$ for all $n > N$ where N is an integer

Example 2: Determine the convergence or divergence of $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{n+1}}$

Example 3: Determine the convergence or divergence of $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2-6n+10}$

Practice 1: Show that the following series converges using the alternating series test

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!} = 1 - \frac{1}{2} + \frac{1}{24} - \frac{1}{720} + \cdots$$

DEFINITION of ABSOLUTE and CONDITIONAL CONVERGENCE

1. $\sum_{n=1}^{\infty} a_n$ is **absolutely convergent** if $\sum_{n=1}^{\infty} |a_n|$ converges.

2. $\sum_{n=1}^{\infty} a_n$ is **conditionally convergent** if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges.

Example 2: The Kitchen Sink of Alternating Series
Determine if the following series are absolutely, conditionally convergent or divergent.

a.) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/2}}$	**b.)** $\sum_{n=1}^{\infty} \frac{(-1)^n}{8}$
c.) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$	**d.)** $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cdot \sqrt[8]{n}}{2n}$

Homework

Do each problem using your notes, then correct your work using my answers and email me you work with corrections.

Consider the geometric series $\sum_{n=0}^{\infty} k\left(\frac{k+3}{6}\right)^n$ where k is a constant.

a.) Find $\sum_{n=0}^{\infty} k\left(\frac{k+3}{6}\right)^n$ when $k = 1$.

b.) Find k when $\sum_{n=0}^{\infty} k\left(\frac{k+3}{6}\right)^n = 12$.

c.) The series $\sum_{n=0}^{\infty} k\left(\frac{k+3}{6}\right)^n$ converges for $a < k < b$ and diverges when $k = a$ or $k = b$. Find a and b.

Practice 3: Each statement below if false. Correct each statement to create a true statement.

For Statements 1 – 3: Let $a_n > 0$

Statement 1: If $a_{n+1} \le a_n$ and $\lim_{n\to\infty} a_n$ converges, then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges

Statement 2: If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges

Statement 3: If $\sum_{n=1}^{\infty} (-1)^n a_n$ diverges, then $\lim_{n\to\infty} a_n = 0$

Statement 4: Consider the series $\sum_{n=1}^{\infty} b_n$. If $\sum_{n=1}^{\infty} b_n$ diverges, then $\lim_{n\to\infty} b_n \ne 0$

Consider the altnerating series defined below:

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

A) Use the alternating series test to show that this series converges when $x = 3$.

AP Practice Problem

Let $a(n) = \frac{1}{n^{k+1}}$ where k is a constant

(**a**) For $k = \frac{1}{2}$, use the alternating series test to show that $\sum_{n=1}^{\infty} (-1)^n a(n)$ converges. Determine if this series converges conditionally or converges absolutely. Explain your reasoning.

(**b**) Let $b(n) = a(\sqrt{n})$. Find all integer values of k such that $\sum_{n=1}^{\infty} (-1)^n b(n)$ converges conditionally.

(**c**) Let $c(n) = a(n^{-2k})$. Show that there is no real value of k such that $\sum_{n=1}^{\infty} c(n)$ is the harmonic series.

AB Test Takers Practice

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Question 4

t (seconds)	0	10	20	30	40	50	60	70	80
$v(t)$ (feet per second)	5	14	22	29	35	40	44	47	49

Rocket A has positive velocity $v(t)$ after being launched upward from an initial height of 0 feet at time $t = 0$ seconds. The velocity of the rocket is recorded for selected values of t over the interval $0 \le t \le 80$ seconds, as shown in the table above.

(a) Find the average acceleration of rocket A over the time interval $0 \le t \le 80$ seconds. Indicate units of measure.

(b) Using correct units, explain the meaning of $\int_{10}^{70} v(t)\,dt$ in terms of the rocket's flight. Use a midpoint Riemann sum with 3 subintervals of equal length to approximate $\int_{10}^{70} v(t)\,dt$.

(c) Rocket B is launched upward with an acceleration of $a(t) = \dfrac{3}{\sqrt{t+1}}$ feet per second per second. At time $t = 0$ seconds, the initial height of the rocket is 0 feet, and the initial velocity is 2 feet per second. Which of the two rockets is traveling faster at time $t = 80$ seconds? Explain your answer.

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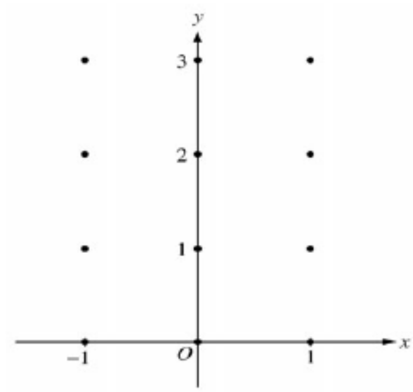
Question 6

Consider the differential equation $\frac{dy}{dx} = x^2(y-1)$.

(a) On the axes provided, sketch a slope field for the given differential equation at the twelve points indicated.
(Note: Use the axes provided in the pink test booklet.)

(b) While the slope field in part (a) is drawn at only twelve points, it is defined at every point in the xy-plane. Describe all points in the xy-plane for which the slopes are positive.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(0) = 3$.



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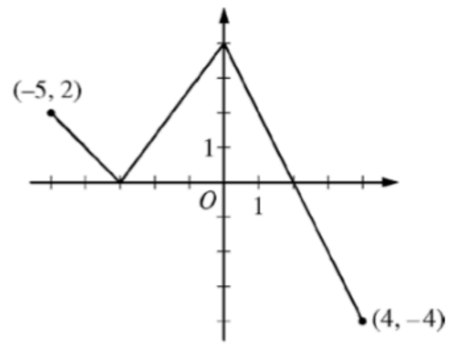
Question 3



Graph of f

The function f is defined on the closed interval $[-5, 4]$. The graph of f consists of three line segments and is shown in the figure above.

Let g be the function defined by $g(x) = \int_{-3}^{x} f(t)\,dt$.

(a) Find $g(3)$.

(b) On what open intervals contained in $-5 < x < 4$ is the graph of g both increasing and concave down? Give a reason for your answer.

(c) The function h is defined by $h(x) = \frac{g(x)}{5x}$. Find $h'(3)$.

(d) The function p is defined by $p(x) = f\left(x^2 - x\right)$. Find the slope of the line tangent to the graph of p at the point where $x = -1$.

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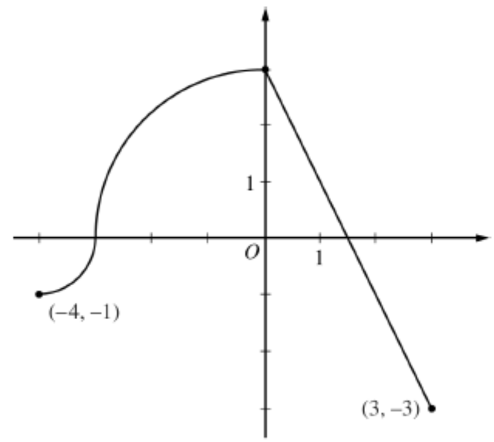
Question 4



Graph of f

The continuous function f is defined on the interval $-4 \le x \le 3$. The graph of f consists of two quarter circles and one line segment, as shown in the figure above.

Let $g(x) = 2x + \int_0^x f(t)\,dt$.

(a) Find $g(-3)$. Find $g'(x)$ and evaluate $g'(-3)$.

(b) Determine the x-coordinate of the point at which g has an absolute maximum on the interval $-4 \le x \le 3$. Justify your answer.

(c) Find all values of x on the interval $-4 < x < 3$ for which the graph of g has a point of inflection. Give a reason for your answer.

(d) Find the average rate of change of f on the interval $-4 \le x \le 3$. There is no point c, $-4 < c < 3$, for which $f'(c)$ is equal to that average rate of change. Explain why this statement does not contradict the Mean Value Theorem.