AP Review 14: Ratio Test and Taylor Series

THE RATIO TEST

Let $\sum_{n=1}^{\infty} a_n$ be a series of nonzero terms.

1. Let $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L,$ a number.

- If $L<1$, then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely.
- If $L=1$, then the ratio test provides no conclusive information about the convergence or divergence of $\sum_{n=1}^{\infty} a_n$.
- If $L>1$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

2. Let $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \Rightarrow \infty,$ then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Example 1: For each of the following series, determine if the series converges or diverges.

a.) $\sum_{n=1}^{\infty} \frac{3^n}{(n-1)!}$	**b.)** $\sum_{n=1}^{\infty} \frac{4^n}{3^{n+1} \cdot n}$	**c.)** $\sum_{n=1}^{\infty} \frac{4^n \cdot n^{10}}{n!}$

Example 2: For each of the following series, find all values of x such that the series converges.

a.) $\sum_{n=1}^{\infty} \frac{(x+3)^n}{n}$	**b.)** $\sum_{n=1}^{\infty} \frac{(2x-4)^n}{n}$

Definitions of nth Taylor Polynomial and nth Macclaurin Polynomial

If f has n derivatives at c, then the polynomial

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)(x-c)^2}{2!} + \cdots + \frac{f^{(n)}(c)(x-c)^n}{n!}$$

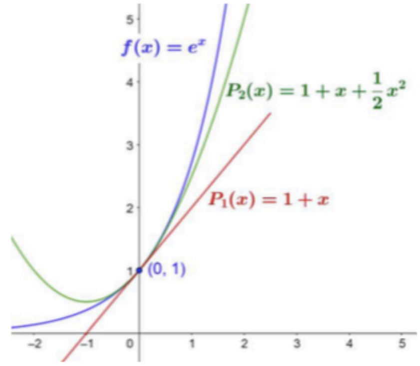
is called the nth **Taylor polynomial** for f at c.

If $c = 0$:

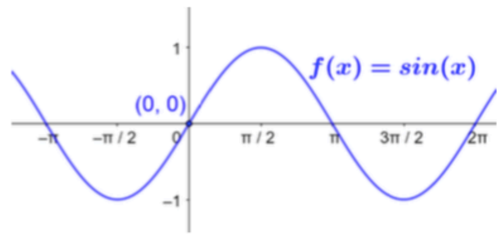
$$P_n(x) = f(0) + f'(0)(x) + \frac{f''(0)(x)^2}{2!} + \cdots + \frac{f^{(n)}(0)(x)^n}{n!}$$

is called the nth **Maclaurin polynomial** for f.

Example 2: Find a second degree polynomial centered at $x = 0$ for $f(x) = e^x$



Example 3: Find a third degree polynomial centered at $x = 0$ for $f(x) = \sin(x)$.
Use this polynomial to approximate $\sin(0.2)$.



x	$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$
1	2	−3	1	−6

Example 5: The functions f and g are differentiable for all orders n. Find a third degree Taylor polynomial for $g(x)$ centered at $x = 1$ where $g(x) = \int_1^x f(t)dt$

Practice 1: Find a second degree Maclaurin polynomial for $f(x) = e^{2x}$.

x	$f(x)$	$f'(x)$	$f''(x)$	$g(x)$	$g'(x)$
3	1	–2	7	4	–5

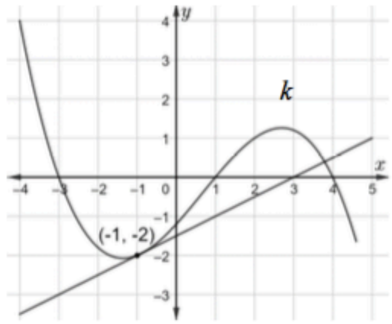
Example 3: The functions f and g are differentiable for all orders. The values of f, g, and selected derivatives of each are given in the table above at $x = 3$. For $n \geq 2$, the nth derivative of g at $x = 3$ is given by $g^{(n)}(3) = f^{(n-2)}(3)$. Find the third degree Taylor polynomial for $g(x)$ about $x = 3$.

Example 4: A function $f(x)$ is not explicitly known but it is known that $f(2) = -7$ and $f'(2) = 0$. Additionally, for $n > 1$, $f^{(n)}(2) = \dfrac{n-1}{3^n}$. Find a 4th degree Taylor polynomial for $f(x)$ centered at $x = 2$. Use this polynomial to approximate $f(3)$.

Homework

Practice 1: The series $\sum_{n=1}^{\infty} a_n (x+2)^n$ diverges when $x = 4$. Determine if the following statements are **A**lways true, **S**ometimes true or **N**ever true

a.) The series diverges when $x = 5$.	**b.)** The series converges when $x = -8$.
c.) The series diverges when $x = 0$.	**d.)** The series converges absolutely when $x = -8$.



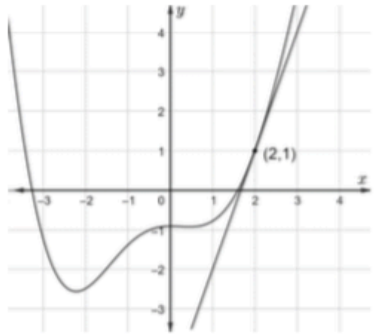
Practice 1: A function k has derivatives of all orders for all values of x. A portion of the graph of k is shown above with the line tangent to the graph of k at $x = -1$. For $n \geq 2$, the nth derivative of $k(x)$ at $x = -1$ is given by:

$$k^{(n)}(-1) = \frac{n!}{n+1}.$$

Find the third degree Taylor polynomial for $k(x)$ about $c = -1$.

Practice 2: The functions f, f', and f'' are each continuous and differentiable. The nth derivative of f is given by $f^{(n)}(1) = \sum_{i=0}^{\infty} 12\left(\frac{n+1}{5}\right)^i$ when $0 \leq n \leq 3$. Find the third degree Taylor polynomial for $f(x)$ centered around $x = 1$.

AP Practice Problem



A function f has derivatives of all orders for all values of x. A portion of the graph of f is shown above with the line tangent to the graph of f at $x = 2$. Let g be the function defined by $g(x) = 3 + \int_2^x f(t)\,dt$.

a.) Find the second degree Taylor polynomial for $g(x)$ centered at $x = 2$.

b.) Does $g(x)$ have a local minimum, local maximum, or neither at $x = 2$? Give a reason for your answer.

c.) Consider the geometric series $\sum_{n=1}^{\infty} a_n$ where $a_1 = g'(2)$ and $a_2 = g'(x) - 1$. Find $\sum_{n=1}^{\infty} a_n$ when $x = \frac{13}{6}$.