AP CALCULUS BC	YouTube Live Virtual Lessons	Mr. Bryan Passwater Mr. Anthony Record
Topic: 6.13	**Arc Length of a Smooth Planar Curve** **Free Response Question Review**	**Date:** April 17, 2020

Arc Length

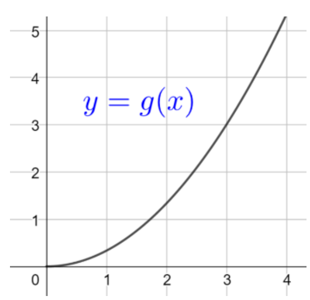
Let the function $y = f(x)$ represent a smooth curve on the interval $[a, b]$, the arc length of f from $x = a$ to $x = b$ is defined by:

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$$

Arc Length Quick Check

Quick Check 1: Consider the function $f(x) = \cos(3x)$. Write, but do not evaluate, an integral expression that gives the arc length of $f(x)$ from 0 to $\frac{2\pi}{3}$

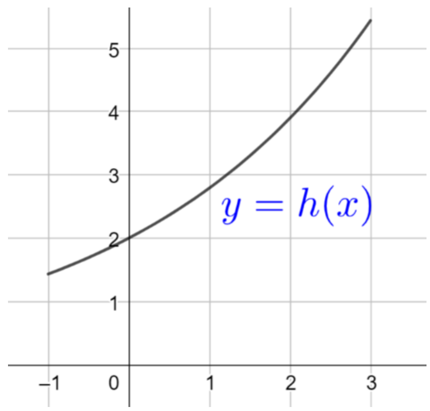
$$L = \int_0^{2\pi/3} \sqrt{1 + \left[-3\sin(3x)\right]^2}\;dx$$



Quick Check 2: A portion of the function $g(x)$ is given above. The arc length of the function $g(x)$ over the interval $[0,3]$ is given by $\int_0^3 \sqrt{1 + \frac{4}{9}x^2}\,dx$. Find $g'(x)$.

$$\left[g'(x)\right]^2 = \frac{4}{9}x^2 \Rightarrow g'(x) = \sqrt{\frac{4}{9}x^2} = \frac{2}{3}x \text{ or } -\frac{2}{3}x.$$

$$g'(x) \neq -\frac{2}{3}x \text{ since } g(x) \text{ is increasing for } x > 0$$



Quick Check 3: A portion of the function $h(x)$ is given above. The arc length of the function $h(x)$ over the interval $[0,2]$ is $\int_0^2 \sqrt{1 + \left[\frac{2}{3} e^{\frac{x}{3}}\right]^2}\, dx$. Find the equation of the tangent line to $h(x)$ at $x = 0$.

$h'(x) = \frac{2}{3} e^{\frac{x}{3}} \Rightarrow h'(0) = \frac{2}{3} e^{\frac{0}{3}} = \frac{2}{3}$ $\qquad$ $T(x) = \frac{2}{3}x + 2$

Quick Check 4: Consider the function $f(x) = \ln(3x^2 + 2)$. The length of the curve of $f(x)$ from $x = 1$ to $x = 4$ is given by $\int_1^4 \sqrt{1 + \frac{ax^2}{(3x^2+2)^2}}\, dx$. Find the value of a.

$L = \int_1^4 \sqrt{1 + \left[f'(x)\right]^2}\, dx$ $\qquad$ $f'(x) = \frac{6x}{(3x^2+2)}$

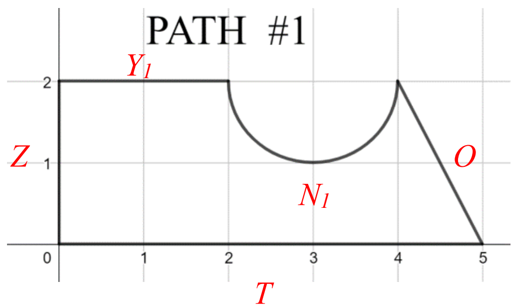
$= \int_1^4 \sqrt{1 + \left[\frac{6x}{(3x^2+2)}\right]^2}\, dx$ $\qquad$ $\left[\frac{6x}{(3x^2+2)}\right]^2 = \frac{ax^2}{(3x^2+2)^2}$ $\qquad$ $36x^2 = ax^2$

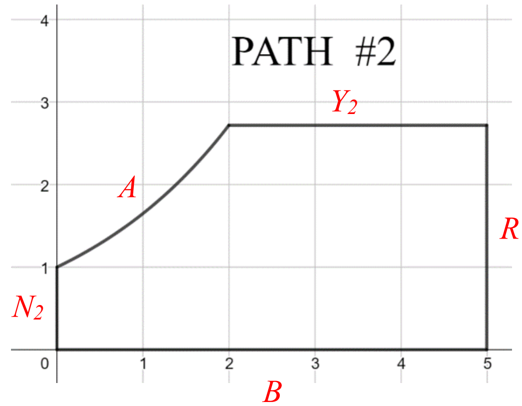
$a = 36$

Quick Check 5: Let $y = f(g(2x))$. Write an integral expression that represents the arc length of y from 0 to 5.

$\frac{dy}{dx} = f'(g(2x))(g'(2x))(2)$ $\qquad$ $L = \int_0^5 \sqrt{1 + \left[2f'(g(2x))(g'(2x))\right]^2}\, dx$

2020 FRQ Practice Problem BC1





BC 1: Tony and Bryan both start at the origin and walk the closed paths as shown above to end back at the origin. Tony follows Path #1 while Bryan follows Path #2. Path #1 consists of four line segments and a semi circle while Path #2 consists of four line segments and the function $f(x) = e^{\frac{x}{2}}$.

(**a**) Who took the longer path, Bryan or Tony?

Path #1: $L_1 = T + O + N_1 + Y_1 + Z$

$$= 5 + \sqrt{2^2 + 1^2} + \left(\frac{1}{2}\pi(2)\right) + 2 + 2 = 9 + \sqrt{5} + \pi = 14.3776\ldots$$

Path #2: $L_2 = B + R + Y_2 + A + N_2$

$$= 5 + e^{\frac{2}{2}} + 3 + \int_0^2 \sqrt{1 + \left[\frac{1}{2}e^{\frac{x}{2}}\right]^2}\,dx + 1 = 9 + e + 2.6625\ldots = 14.3808\ldots$$

Bryan's path was longer.

(**b**) Whose path created the region with the largest area?

Path #1 area: $(2)(2) + \left[4 - \frac{1}{2}\pi(1)^2\right] + \frac{1}{2}(1)(2) = 9 - \frac{1}{2}\pi = 7.4292\ldots$

Path #2 area: $\int_0^2 e^{\frac{x}{2}}\,dx + (3)(e) = 3e + 2\left[e^{\frac{x}{2}}\right]_0^2 = 3e + 2[e - 1] = 5e - 2 = 11.5914\ldots$

Bryan's path's area is largest

(**c**) Let $P_2(x)$ be the second degree Maclaurin polynomial for $f(x)$. Find $P_2(x)$ and use it to approximate $f(2)$.

$$P_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2$$

$$f(0) = 1$$

$$f'(0) = \frac{1}{2}e^{\frac{0}{2}} = \frac{1}{2}$$

$$f''(0) = \frac{1}{4}e^{\frac{0}{2}} = \frac{1}{4}$$

$$P_2(x) = 1 + \frac{1}{2}x + \frac{1/4}{2!}x^2 = 1 + \frac{1}{2}x + \frac{1}{8}x^2$$

$$f(2) \approx P_2(2) = 1 + \frac{1}{2}(2) + \frac{1}{8}(2)^2 = \frac{5}{2}$$

2020 FRQ Practice Problem BC2

BC 2: Let g be the function defined by $g(x) = \begin{cases} a\sin(x^2-1), & 0 \le x \le 1 \\ \ln(x), & x > 1 \end{cases}$ where a is a constant.

(**a**) Show that $g(x)$ is continuous at $x = 1$.

$$\lim_{x\to 1^-} g(x) = \lim_{x\to 1^-}\left[a\sin\left(x^2-1\right)\right] = \left[a\sin\left((1)^2-1\right)\right] = \left[a\sin(0)\right] = 0$$

$$\lim_{x\to 1^+} g(x) = \lim_{x\to 1^+}\left[\ln(x)\right] = \left[\ln(1)\right] = 0 \qquad g(1) = a\sin\left((1)^2-1\right) = 0$$

$g(x)$ is continuous at $x = 1$ because $g(1) = \lim_{x\to 1} g(x)$

(**b**) Find the value of a that guarantees the function $g(x)$ is differentiable at $x = 1$.

$$g'(x) = \begin{cases} a\cos\left(x^2-1\right)(2x), & 0 \le x \le 1 \\ \dfrac{1}{x}, & x > 1 \end{cases}$$

$$\lim_{x\to 1^-} g'(x) = \lim_{x\to 1^+} g'(x) \Rightarrow \lim_{x\to 1^-}\left[a\cos\left(x^2-1\right)(2x)\right] = \lim_{x\to 1^+}\left[\frac{1}{x}\right]$$

$$\lim_{x\to 1^-}\left[a\cos\left(x^2-1\right)(2x)\right] = a\cos(0)(2) = 2a \qquad \lim_{x\to 1^+}\left[\frac{1}{x}\right] = 1 \Rightarrow 2a = 1 \Rightarrow a = \frac{1}{2}$$

(**c**) Write, but do not evaluate, an expression involving one or more integrals that gives the arc length of $g(x)$ from $x = 0$ to $x = 3$.

$$L = \int_0^1 \sqrt{1 + \left[x\cos\left(x^2-1\right)\right]^2}\,dx + \int_1^3 \sqrt{1 + \left[\frac{1}{x}\right]^2}\,dx$$

2020 FRQ Practice Problem BC3

BC 3: Consider the continuous and differentiable positive function f such that the arc length of the curve f from 0 to 2 is equal to the area bounded between the graph of f and the x axis from 0 to 2.

(**a**) Show that the function $y = f(x)$ satisfies the differential equation $y^2 - (y')^2 - 1 = 0$.

$$\int_0^2 \sqrt{1+\left[f'(x)\right]^2}\ dx = \int_0^2 f(x)dx \Rightarrow \sqrt{1+\left[f'(x)\right]^2} = f(x)$$

$$\left(f(x)\right)^2 - \left(f'(x)\right)^2 - 1 = \left(1+\left[f'(x)\right]^2\right) - \left(f'(x)\right)^2 - 1 = 0$$

$$y^2 \quad - \quad (y')^2 \quad -1 \qquad\qquad\qquad\qquad = 0$$

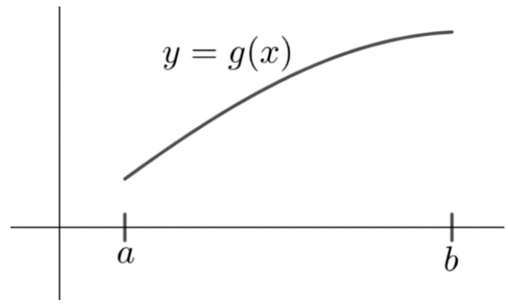
(**b**) One such function that satisfies the differential equation in part (a) is $g(x) = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$.

Find the arc length of $g(x)$ from $x = 0$ to $x = 2$.

$$L = \int_0^2 \sqrt{1+\left[g'(x)\right]^2}\ dx = \int_0^2 g(x)dx = \int_0^2 \left(\frac{1}{2}e^x + \frac{1}{2}e^{-x}\right)dx$$

$$= \frac{1}{2}\left[e^x - e^{-x}\right]_0^2 = \frac{1}{2}\left[\left(e^2 - e^{-2}\right) - \left(e^0 - e^{-0}\right)\right] = \frac{1}{2}\left[\left(e^2 - e^{-2}\right)\right]$$

2020 FRQ Practice Problem BC4



BC 4: A portion of the graph for the differentiable function g is given above for $a \leq x \leq b$.

The arc length of g over the interval $[a, b]$ is equal to L where L $= \int_a^b \sqrt{1 + 4\cos^2(2x)}\, dx$.

(**a**) Find $g'(0)$

$$\left[g'(x)\right]^2 = 4\cos^2(2x) \Rightarrow g'(x) = 2\cos(2x) \Rightarrow g'(0) = 2\cos(0) = 2$$

(**b**) Find $\int g(x)dx$

$$g'(x) = 2\cos(2x) \Rightarrow g(x) = \sin(2x) + C$$

$$\int \sin(2x)dx \quad = -\frac{1}{2}\cos(2x) + Cx + D$$

(**c**) Find the equation of the line tangent to $g(x)$ at $x = \frac{\pi}{6}$

$$T(x) = g\left(\frac{\pi}{6}\right) + g'\left(\frac{\pi}{6}\right)\left(x - \frac{\pi}{6}\right)$$

$$g\left(\frac{\pi}{6}\right) = \sin\left(\frac{\pi}{3}\right) + C = \frac{\sqrt{3}}{2} + C$$

$$g'\left(\frac{\pi}{6}\right) = 2\cos\left(\frac{\pi}{3}\right) = (2)\left(\frac{1}{2}\right) = 1$$

$$T(x) = \left(\frac{\sqrt{3}}{2} + C\right) + (1)\left(x - \frac{\pi}{6}\right) = \left(\frac{\sqrt{3}}{2}\right) + \left(x - \frac{\pi}{6}\right) + C$$

2020 FRQ Practice Problem BC5

x	0	1	4	6	8
$f'(x)$	$\sqrt{8}$	$\sqrt{3}$	0	$\sqrt{3}$	2

BC 5: The function f is twice differentiable for all real values with $f''(0) = -\frac{3}{8\sqrt{2}}$. Selected values of f', the derivative of f, are given in the table above. The arc length of the function $f(x)$ from 0 to x can be represented by the function S, defined by $S(x) = \int_0^x \sqrt{1 + [f'(t)]^2}\,dt$.

(**a**) Using a left Riemann sum with the four subintervals indicated in the table, approximate the arc length of the function $f(x)$ from $x = 0$ to $x = 8$.

$$\int_0^8 \sqrt{1+[f'(x)]^2}\,dx \approx 15$$

$$\approx (1)\left(\sqrt{1+[f'(0)]^2}\right)+(3)\left(\sqrt{1+[f'(1)]^2}\right)+(2)\left(\sqrt{1+[f'(4)]^2}\right)+(2)\left(\sqrt{1+[f'(6)]^2}\right)$$

$$= \left(\sqrt{9}\right) +(3) \left(\sqrt{4}\right) +(2) \left(\sqrt{1}\right) +(2) \left(\sqrt{4}\right) = 15$$

(**b**) Use Euler's method, starting at $x = 0$ with two steps of equal size to approximate $S(8)$. Show the work that leads to your answer.

$$S(x)=\int_0^x \sqrt{1+[f'(t)]^2}\,dt \Rightarrow S'(x)=\sqrt{1+[f'(x)]^2}$$

x	$S(x)$	$S'(x)dx$
0	0	$\sqrt{1+[f'(0)]^2}(4)=\sqrt{1+[\sqrt{8}]^2}(4)=12$
4	12	$\sqrt{1+[f'(4)]^2}(4)=\sqrt{1+[0]^2}(4)=4$
8	16	

$S(8) \approx 8$

(**c**) Let $P_2(x)$ be the second degree Maclaurin polynomial for $S(x)$. Find $P_2(x)$ and use it to approximate $S(8)$.

$$S(0)=0$$

$$P_2(x)=S(0)+S'(0)x+\frac{S''(0)}{2!}x^2 \qquad S'(0)=\sqrt{1+\left[f'(0)\right]^2}=3$$

$$S''(0)=\frac{2f'(0)f''(0)}{2\sqrt{1+\left[f'(0)\right]^2}}=\frac{\left(\sqrt{8}\right)\left(-\frac{3}{8\sqrt{2}}\right)}{3}=\frac{-(2)}{8}=-\frac{1}{4}$$

$$P_2(x)=3x-\frac{1}{8}x^2 \qquad S(8)\approx 3(8)-\frac{1}{8}(8)^2=24-8=16$$

Question 5

A population is modeled by a function P that satisfies the logistic differential equation

$$\frac{dP}{dt} = \frac{P}{5}\left(1 - \frac{P}{12}\right).$$

(a) If $P(0) = 3,$ what is $\lim_{t \to \infty} P(t)$?

If $P(0) = 20,$ what is $\lim_{t \to \infty} P(t)$?

(b) If $P(0) = 3,$ for what value of P is the population growing the fastest?

(c) A different population is modeled by a function Y that satisfies the separable differential equation

$$\frac{dY}{dt} = \frac{Y}{5}\left(1 - \frac{t}{12}\right).$$

Find $Y(t)$ if $Y(0) = 3.$

(d) For the function Y found in part (c), what is $\lim_{t \to \infty} Y(t)$?

(a) For this logistic differential equation, the carrying capacity is 12. If $P(0) = 3,\ \lim_{t \to \infty} P(t) = 12.$ If $P(0) = 20,\ \lim_{t \to \infty} P(t) = 12.$	$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{answer} \end{cases}$
(b) The population is growing the fastest when P is half the carrying capacity. Therefore, P is growing the fastest when $P = 6.$	1 : answer
(c) $\frac{1}{Y}dY = \frac{1}{5}\left(1 - \frac{t}{12}\right)dt = \left(\frac{1}{5} - \frac{t}{60}\right)dt$ $\ln\lvert Y\rvert = \frac{t}{5} - \frac{t^2}{120} + C$ $Y(t) = Ke^{\frac{t}{5} - \frac{t^2}{120}}$ $K = 3$ $Y(t) = 3e^{\frac{t}{5} - \frac{t^2}{120}}$	$5 : \begin{cases} 1 : \text{separates variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } Y \\ \quad 0/1 \text{ if } Y \text{ is not exponential} \end{cases}$ Note: max $2/5$ [1-1-0-0-0] if no constant of integration Note: $0/5$ if no separation of variables
(d) $\lim_{t \to \infty} Y(t) = 0$	1 : answer $0/1$ if Y is not exponential

Question 6

Let f be the function given by $f(x) = \sin\left(5x + \frac{\pi}{4}\right)$, and let $P(x)$ be the third-degree Taylor polynomial for f about $x = 0$.

(a) Find $P(x)$.

(b) Find the coefficient of x^{22} in the Taylor series for f about $x = 0$.

(c) Use the Lagrange error bound to show that $\left| f\left(\frac{1}{10}\right) - P\left(\frac{1}{10}\right) \right| < \frac{1}{100}$.

(d) Let G be the function given by $G(x) = \int_0^x f(t)\,dt$. Write the third-degree Taylor polynomial for G about $x = 0$.

(a) $f(0) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$f'(0) = 5\cos\left(\frac{\pi}{4}\right) = \frac{5\sqrt{2}}{2}$

$f''(0) = -25\sin\left(\frac{\pi}{4}\right) = -\frac{25\sqrt{2}}{2}$

$f'''(0) = -125\cos\left(\frac{\pi}{4}\right) = -\frac{125\sqrt{2}}{2}$

$P(x) = \frac{\sqrt{2}}{2} + \frac{5\sqrt{2}}{2}x - \frac{25\sqrt{2}}{2(2!)}x^2 - \frac{125\sqrt{2}}{2(3!)}x^3$

4 : $P(x)$

$\langle -1 \rangle$ each error or missing term

deduct only once for $\sin\left(\frac{\pi}{4}\right)$ evaluation error

deduct only once for $\cos\left(\frac{\pi}{4}\right)$ evaluation error

$\langle -1 \rangle$ max for all extra terms, $+\cdots$, misuse of equality

(b) $\frac{-5^{22}\sqrt{2}}{2(22!)}$

2 : { 1 : magnitude; 1 : sign }

(c) $\left| f\left(\frac{1}{10}\right) - P\left(\frac{1}{10}\right) \right| \le \max_{0 \le c \le \frac{1}{10}} \left| f^{(4)}(c) \right| \left(\frac{1}{4!}\right)\left(\frac{1}{10}\right)^4$

$\le \frac{625}{4!}\left(\frac{1}{10}\right)^4 = \frac{1}{384} < \frac{1}{100}$

1 : error bound in an appropriate inequality

(d) The third-degree Taylor polynomial for G about $x = 0$ is $\int_0^x \left(\frac{\sqrt{2}}{2} + \frac{5\sqrt{2}}{2}t - \frac{25\sqrt{2}}{4}t^2 \right) dt$

$= \frac{\sqrt{2}}{2}x + \frac{5\sqrt{2}}{4}x^2 - \frac{25\sqrt{2}}{12}x^3$

2 : third-degree Taylor polynomial for G about $x = 0$

$\langle -1 \rangle$ each incorrect or missing term

$\langle -1 \rangle$ max for all extra terms, $+\cdots$, misuse of equality