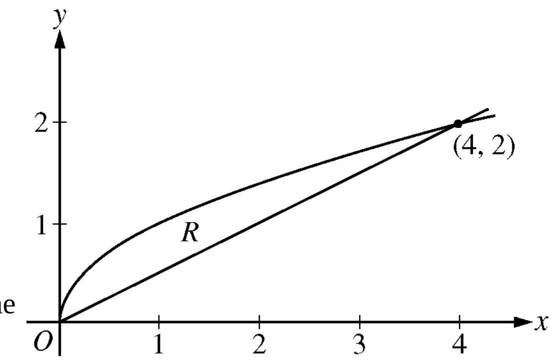
AP® CALCULUS AB
2009 SCORING GUIDELINES (Form B)

Question 4

Let R be the region bounded by the graphs of $y = \sqrt{x}$ and $y = \frac{x}{2}$, as shown in the figure above.



(a) Find the area of R.

(b) The region R is the base of a solid. For this solid, the cross sections perpendicular to the x-axis are squares. Find the volume of this solid.

(c) Write, but do not evaluate, an integral expression for the volume of the solid generated when R is rotated about the horizontal line $y = 2$.

(a) Area $= \int_0^4 \left(\sqrt{x} - \frac{x}{2}\right) dx = \frac{2}{3}x^{3/2} - \frac{x^2}{4}\Big\vert_{x=0}^{x=4} = \frac{4}{3}$	3 : { 1 : integrand 1 : antiderivative 1 : answer
(b) Volume $= \int_0^4 \left(\sqrt{x} - \frac{x}{2}\right)^2 dx = \int_0^4 \left(x - x^{3/2} + \frac{x^2}{4}\right) dx$ $= \frac{x^2}{2} - \frac{2x^{5/2}}{5} + \frac{x^3}{12}\Big\vert_{x=0}^{x=4} = \frac{8}{15}$	3 : { 1 : integrand 1 : antiderivative 1 : answer
(c) Volume $= \pi \int_0^4 \left(\left(2 - \frac{x}{2}\right)^2 - \left(2 - \sqrt{x}\right)^2\right) dx$	3 : { 1 : limits and constant 2 : integrand

Question 4

Consider the differential equation $\frac{dy}{dx} = 2x - y$.

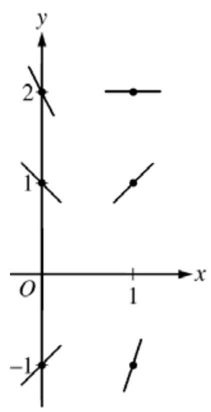
(a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.

(b) Find $\frac{d^2y}{dx^2}$ in terms of x and y. Determine the concavity of all solution curves for the given differential equation in Quadrant II. Give a reason for your answer.

(c) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(2) = 3$. Does f have a relative minimum, a relative maximum, or neither at $x = 2$? Justify your answer.

(d) Find the values of the constants m and b for which $y = mx + b$ is a solution to the differential equation.

(a)



$2 : \begin{cases} 1 : \text{slopes where } x = 0 \\ 1 : \text{slopes where } x = 1 \end{cases}$

(b) $\frac{d^2y}{dx^2} = 2 - \frac{dy}{dx} = 2 - (2x - y) = 2 - 2x + y$

In Quadrant II, $x < 0$ and $y > 0$, so $2 - 2x + y > 0$.
Therefore, all solution curves are concave up in Quadrant II.

$2 : \begin{cases} 1 : \frac{d^2y}{dx^2} \\ 1 : \text{concave up with reason} \end{cases}$

(c) $\left.\frac{dy}{dx}\right|_{(x,\, y)=(2,\, 3)} = 2(2) - 3 = 1 \neq 0$

Therefore, f has neither a relative minimum nor a relative maximum at $x = 2$.

$2 : \begin{cases} 1 : \text{considers } \left.\frac{dy}{dx}\right|_{(x,\, y)=(2,\, 3)} \\ 1 : \text{conclusion with justification} \end{cases}$

(d) $y = mx + b \Rightarrow \frac{dy}{dx} = \frac{d}{dx}(mx + b) = m$

$2x - y = m$

$2x - (mx + b) = m$

$(2 - m)x - (m + b) = 0$

$2 - m = 0 \Rightarrow m = 2$

$b = -m \Rightarrow b = -2$

Therefore, $m = 2$ and $b = -2$.

$3 : \begin{cases} 1 : \frac{d}{dx}(mx + b) = m \\ 1 : 2x - y = m \\ 1 : \text{answer} \end{cases}$

AP® CALCULUS BC
2007 SCORING GUIDELINES

Question 4

Let f be the function defined for $x > 0$, with $f(e) = 2$ and f', the first derivative of f, given by $f'(x) = x^2 \ln x$.

(a) Write an equation for the line tangent to the graph of f at the point $(e, 2)$.

(b) Is the graph of f concave up or concave down on the interval $1 < x < 3$? Give a reason for your answer.

(c) Use antidifferentiation to find $f(x)$.

(a) $f'(e) = e^2$

An equation for the line tangent to the graph of f at the point $(e, 2)$ is $y - 2 = e^2(x - e)$.

$2 : \begin{cases} 1 : f'(e) \\ 1 : \text{equation of tangent line} \end{cases}$

(b) $f''(x) = x + 2x \ln x.$

For $1 < x < 3$, $x > 0$ and $\ln x > 0$, so $f''(x) > 0$. Thus, the graph of f is concave up on $(1, 3)$.

$3 : \begin{cases} 2 : f''(x) \\ 1 : \text{answer with reason} \end{cases}$

(c) Since $f(x) = \int \left(x^2 \ln x\right) dx$, we consider integration by parts.

$$u = \ln x \qquad dv = x^2\, dx$$

$$du = \frac{1}{x}\, dx \qquad v = \int \left(x^2\right) dx = \frac{1}{3}x^3$$

Therefore,

$$\begin{aligned} f(x) &= \int \left(x^2 \ln x\right) dx \\ &= \frac{1}{3}x^3 \ln x - \int \left(\frac{1}{3}x^3 \cdot \frac{1}{x}\right) dx \\ &= \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C. \end{aligned}$$

Since $f(e) = 2$, $2 = \frac{e^3}{3} - \frac{e^3}{9} + C$ and $C = 2 - \frac{2}{9}e^3$.

Thus, $f(x) = \frac{x^3}{3} \ln x - \frac{1}{9}x^3 + 2 - \frac{2}{9}e^3$.

$4 : \begin{cases} 2 : \text{antiderivative} \\ 1 : \text{uses } f(e) = 2 \\ 1 : \text{answer} \end{cases}$

Question 5

Consider the differential equation $\frac{dy}{dx} = 5x^2 - \frac{6}{y-2}$ for $y \neq 2$. Let $y = f(x)$ be the particular solution to this differential equation with the initial condition $f(-1) = -4$.

(a) Evaluate $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $(-1, -4)$.

(b) Is it possible for the x-axis to be tangent to the graph of f at some point? Explain why or why not.

(c) Find the second-degree Taylor polynomial for f about $x = -1$.

(d) Use Euler's method, starting at $x = -1$ with two steps of equal size, to approximate $f(0)$. Show the work that leads to your answer.

(a) $\left.\frac{dy}{dx}\right|_{(-1,-4)} = 6$

$$\frac{d^2y}{dx^2} = 10x + 6(y-2)^{-2}\frac{dy}{dx}$$

$$\left.\frac{d^2y}{dx^2}\right|_{(-1,-4)} = -10 + 6\frac{1}{(-6)^2}6 = -9$$

$3: \begin{cases} 1: \left.\frac{dy}{dx}\right|_{(-1,-4)} \\ 1: \frac{d^2y}{dx^2} \\ 1: \left.\frac{d^2y}{dx^2}\right|_{(-1,-4)} \end{cases}$

(b) The x-axis will be tangent to the graph of f if $\left.\frac{dy}{dx}\right|_{(k,0)} = 0$.

The x-axis will never be tangent to the graph of f because $\left.\frac{dy}{dx}\right|_{(k,0)} = 5k^2 + 3 > 0$ for all k.

$2: \begin{cases} 1: \frac{dy}{dx} = 0 \text{ and } y = 0 \\ 1: \text{answer and explanation} \end{cases}$

(c) $P(x) = -4 + 6(x+1) - \frac{9}{2}(x+1)^2$

$2: \begin{cases} 1: \text{quadratic and centered at } x = -1 \\ 1: \text{coefficients} \end{cases}$

(d) $f(-1) = -4$

$$f\left(-\frac{1}{2}\right) \approx -4 + \frac{1}{2}(6) = -1$$

$$f(0) \approx -1 + \frac{1}{2}\left(\frac{5}{4} + 2\right) = \frac{5}{8}$$

$2: \begin{cases} 1: \text{Euler's method with 2 steps} \\ 1: \text{Euler's approximation to } f(0) \end{cases}$