Question 4

Solution	Scoring
(a) $V = \pi r^2 h = \pi (1)^2 h = \pi h$ $\left.\frac{dV}{dt}\right\|_{h=4} = \pi \left.\frac{dh}{dt}\right\|_{h=4} = \pi\left(-\frac{1}{10}\sqrt{4}\right) = -\frac{\pi}{5}$ cubic feet per second	2 : { 1 : $\frac{dV}{dt} = \pi \frac{dh}{dt}$ 1 : answer with units
(b) $\frac{d^2h}{dt^2} = -\frac{1}{20\sqrt{h}} \cdot \frac{dh}{dt} = -\frac{1}{20\sqrt{h}} \cdot \left(-\frac{1}{10}\sqrt{h}\right) = \frac{1}{200}$ Because $\frac{d^2h}{dt^2} = \frac{1}{200} > 0$ for $h > 0$, the rate of change of the height is increasing when the height of the water is 3 feet.	3 : { 1 : $\frac{d}{dh}\left(-\frac{1}{10}\sqrt{h}\right) = -\frac{1}{20\sqrt{h}}$ 1 : $\frac{d^2h}{dt^2} = -\frac{1}{20\sqrt{h}} \cdot \frac{dh}{dt}$ 1 : answer with explanation
(c) $\frac{dh}{\sqrt{h}} = -\frac{1}{10}\, dt$ $\int \frac{dh}{\sqrt{h}} = \int -\frac{1}{10}\, dt$ $2\sqrt{h} = -\frac{1}{10}t + C$ $2\sqrt{5} = -\frac{1}{10} \cdot 0 + C \Rightarrow C = 2\sqrt{5}$ $2\sqrt{h} = -\frac{1}{10}t + 2\sqrt{5}$ $h(t) = \left(-\frac{1}{20}t + \sqrt{5}\right)^2$	4 : { 1 : separation of variables 1 : antiderivatives 1 : constant of integration and uses initial condition 1 : $h(t)$ Note: $0/4$ if no separation of variables Note: max $2/4$ [1-1-0-0] if no constant of integration

Question 5

(a) $f'(x) = \dfrac{-(2x-2)}{\left(x^2 - 2x + k\right)^2}$

$f'(0) = \dfrac{2}{k^2} = 6 \Rightarrow k^2 = \dfrac{1}{3} \Rightarrow k = \dfrac{1}{\sqrt{3}}$

3 : $\begin{cases} 1 : \text{denominator of } f'(x) \\ 1 : f'(x) \\ 1 : \text{answer} \end{cases}$

(b) $\dfrac{1}{x^2 - 2x - 8} = \dfrac{1}{(x-4)(x+2)} = \dfrac{A}{x-4} + \dfrac{B}{x+2}$

$\Rightarrow 1 = A(x+2) + B(x-4)$

$\Rightarrow A = \dfrac{1}{6},\ B = -\dfrac{1}{6}$

$$\int_0^1 f(x)\,dx = \int_0^1 \left(\frac{\frac{1}{6}}{x-4} - \frac{\frac{1}{6}}{x+2} \right) dx$$

$$= \left[\frac{1}{6}\ln|x-4| - \frac{1}{6}\ln|x+2| \right]_{x=0}^{x=1}$$

$$= \left(\frac{1}{6}\ln 3 - \frac{1}{6}\ln 3 \right) - \left(\frac{1}{6}\ln 4 - \frac{1}{6}\ln 2 \right) = -\frac{1}{6}\ln 2$$

3 : $\begin{cases} 1 : \text{partial fraction decomposition} \\ 1 : \text{antiderivatives} \\ 1 : \text{answer} \end{cases}$

(c) $$\int_0^2 \frac{1}{x^2 - 2x + 1}\,dx = \int_0^2 \frac{1}{(x-1)^2}\,dx = \int_0^1 \frac{1}{(x-1)^2}\,dx + \int_1^2 \frac{1}{(x-1)^2}\,dx$$

$$= \lim_{b \to 1^-} \int_0^b \frac{1}{(x-1)^2}\,dx + \lim_{b \to 1^+} \int_b^2 \frac{1}{(x-1)^2}\,dx$$

$$= \lim_{b \to 1^-} \left(-\frac{1}{x-1} \Bigg|_{x=0}^{x=b} \right) + \lim_{b \to 1^+} \left(-\frac{1}{x-1} \Bigg|_{x=b}^{x=2} \right)$$

$$= \lim_{b \to 1^-} \left(-\frac{1}{b-1} - 1 \right) + \lim_{b \to 1^+} \left(-1 + \frac{1}{b-1} \right)$$

Because $\lim\limits_{b \to 1^-} \left(-\dfrac{1}{b-1} \right)$ does not exist, the integral diverges.

3 : $\begin{cases} 1 : \text{improper integral} \\ 1 : \text{antiderivative} \\ 1 : \text{answer with reason} \end{cases}$

Question 6

(a) $f(0) = 3$ and $f'(0) = -2$

The third-degree Taylor polynomial for f about $x = 0$ is

$$3 - 2x + \frac{3}{2!}x^2 + \frac{-\frac{23}{2}}{3!}x^3 = 3 - 2x + \frac{3}{2}x^2 - \frac{23}{12}x^3.$$

$2 : \begin{cases} 1 : \text{two terms} \\ 1 : \text{remaining terms} \end{cases}$

(b) The first three nonzero terms of the Maclaurin series for e^x are $1 + x + \frac{1}{2!}x^2$.

The second-degree Taylor polynomial for $e^x f(x)$ about $x = 0$ is

$$3\left(1 + x + \frac{1}{2!}x^2\right) - 2x(1 + x) + \frac{3}{2}x^2(1)$$
$$= 3 + (3 - 2)x + \left(\frac{3}{2} - 2 + \frac{3}{2}\right)x^2$$
$$= 3 + x + x^2.$$

$2 : \begin{cases} 1 : \text{three terms for } e^x \\ 1 : \text{three terms for } e^x f(x) \end{cases}$

(c) $h(1) = \int_0^1 f(t)\,dt$

$$\approx \int_0^1 \left(3 - 2t + \frac{3}{2}t^2 - \frac{23}{12}t^3\right) dt$$
$$= \left[3t - t^2 + \frac{1}{2}t^3 - \frac{23}{48}t^4\right]_{t=0}^{t=1}$$
$$= 3 - 1 + \frac{1}{2} - \frac{23}{48} = \frac{97}{48}$$

$2 : \begin{cases} 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

(d) The alternating series error bound is the absolute value of the first omitted term of the series for $h(1)$.

$$\int_0^1 \left(\frac{54}{4!}t^4\right) dt = \left[\frac{9}{20}t^5\right]_{t=0}^{t=1} = \frac{9}{20}$$

$$\text{Error} \le \left|\frac{9}{20}\right| = 0.45$$

$3 : \begin{cases} 1 : \text{uses fourth-degree term of Maclaurin series for } f \\ 1 : \text{uses first omitted term of series for } h(1) \\ 1 : \text{error bound} \end{cases}$