I.9 Taylor Series and Approximations

Why do we even care about Taylor Series?

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

So can we approximate other functions with series?

Let's make a power series (a Taylor Series) for sin(x) centered at x=0

Definitions of nth Taylor Polynomial and nth Macclaurin Polynomial

If f has n derivatives at c, then the polynomial

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)(x-c)^2}{2!} + \cdots + \frac{f^{(n)}(c)(x-c)^n}{n!}$$

is called the nth **Taylor polynomial** for f at c.

If $c = 0$:

$$P_n(x) = f(0) + f'(0)(x) + \frac{f''(0)(x)^2}{2!} + \cdots + \frac{f^{(n)}(0)(x)^n}{n!}$$

is called the nth **Maclaurin polynomial** for f.

Practice 1: Find a second degree Maclaurin polynomial for $f(x) = e^{2x}$.

Practice 2: Find a third degree Taylor polynomial for $g(x) = ln(x)$ centered at $x = 1$.

Practice 3: A function h has derivatives of all orders at $x = 0$. It is known that $h(0) = 2, h'(0) = -1$, and $h^{(n)}(0) = \frac{n^2}{n!}$ for $n \geq 2$. Find a third degree Maclaurin Polynomial for $h(x)$.

x	1	2
$f(x)$	–2	0
$f'(x)$	3	–1
$f''(x)$	5	–6

Example 5: The function h is defined by $h(x) = 4 + \int_{2}^{2x} f(t)\,dt$ where f is a twice differentiable function. Selected values of f and its derivatives are given in the table above. Find the 2nd degree Taylor polynomial for $h(x)$ centered at $x = 1$.

Practice 3: The function g is continuous and has derivatives for all orders at $x = -1$. It is known that $g(-1) = 7$ and for positive values of n, the nth derivative of g at $x = -1$ is defined as the piecewise function given below:

$$g^{(n)}(-1) = \begin{cases} n^2 + 1, & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases}$$

a.) Find $P_5(x)$, the fifth degree Taylor polynomial of $g(x)$ centered at $x = -1$.

b.) Determine if $P_5(x)$ is increasing or decreasing at $x = -1$. Explain your reasoning.

Practice 4: The fourth degree Taylor polynomial for $f(x)$ centered about $x = 2$ is given by

$T_4(x) = 2 - 3(x-2) + \frac{3(x-2)^2}{4} - \frac{4(x-2)^3}{9} + \frac{7(x-2)^4}{26}$. Find the value of $f'''(2)$.

A function f has derivatives of all orders at all real x values.

a.) Let $P_2(x)$ represent the 2nd degree Maclaurin polynomial for f. It is known that $f(0)=1$ and $f'(0)=0$. If $P_2(1)=\frac{1}{2}$, find $f''(0)$.

b.) Find $P_2'(0)$ and $P_2''(0)$. Does $P_2(x)$ have a relative minimum, relative maximum, or neither at $x=0$? Give a reason for your answer.

c.) Use $P_2(x)$ to approximate $f\left(\frac{1}{2}\right)$.

when you finally realize Napoleon Dynamite is actually Taylor Swift in drag