10.7 Taylor Series

By now, you are probably asking, "why are we learning this series stuff?" Starting today, we start seeing why! While the curtain will not be all the way revealed, we will start pulling it back to see the connections.

Let's start with something from early in the unit. Recall the following:

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \text{ if } |r|<1$$

For the sake of argument, let's say a=1 and $r = x$. Then, we could say:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \text{ if } |x|<1 \qquad \text{Equation 1}$$

Well, we could understand the ____________________________ $\sum_{n=0}^{\infty} x^n$ as representing the function:

$$f(x) = \frac{1}{1-x} \qquad \text{Equation 2}$$

if $|x|<1$.

We need to remember that condition. Any value $x \neq 1$ will work for f(x), but only ______ will make the power series converge. For

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

the condition $|x|<1$ has to be present. What this means is that the ____________________________ is R=1 and the _____________________________ is $|x|<1$.

Power Series

$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$ is a ____________________ centered at ______.

$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots$ is a power series centered at x=a.

This all might seem trivial, but it is revolutionary. What we are saying is that a function can actually be represented by a relatively simple, infinite polynomial. For example:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots$$

within some interval of x-values.

WOOOOOOAAAAAAAHHHHHH…

Example: Find a power series representation for the following function and determine its interval of convergence.

$$f(x) = \frac{1}{1+x}$$

Practice: Find a power series representation for the following function and determine its interval of convergence.

$$f(x) = \frac{1}{1+x^3}$$

Example: Given that $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots$ on the interval (-1, 1), find a power series to represent $\frac{1}{(1-x)^2}$.

Example: Given that $\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$ on the interval (-1, 1), find a power series to represent $\ln(1 + x)$.

A few observations: if we perform _____________ differentiation or integration on some series $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots$ that converges for $|x - a| < R$, the resulting series represents $f'(x)$ or $\int_a^x f(t)\,dt$ and converges like the original series for $|x - a| < R$.

Taylor Series

It would be annoying to have to always come up with some variation on $f(x) = \frac{1}{1-x}$ to determine a power series for a function. Fortunately, there are ______________________.

Let f be a function with derivatives of all orders throughout some open interval containing a. The the **Taylor series** that is generated by f at $x=a$ is

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

The partial sum

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

is the **Taylor polynomial** of order n for f at $x=a$.

When a Taylor series or polynomial is centered at $x=0$, it is called a **Maclaurin series or polynomial,** respectively.

Example: Construct the seventh order Taylor polynomial for sin(x) at x=0.

Practice: Construct the fourth order Taylor polynomial for e^x at x=2.

Important Maclaurin series: $e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

Example: Find a Maclaurin series to represent $f(x) = \frac{e^{2x}-1}{4}$.

Important Maclaurin Series to Memorize

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 \cdots = \sum_{n=0}^{\infty} x^n$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$$

Homework: Textbook pg. 500 #1-4, 5, 13, 19