10.8 Working with Taylor Series

The root of everything we will be doing comes from the formula for Taylor series. Memorize it!

Let f be a function with derivatives of all orders throughout some open interval containing a. The the **Taylor series** that is generated by f at x=a is

$$f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n+\cdots=\sum_{k=0}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k$$

The partial sum

$$P_n(x)=\sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k$$

is the **Taylor polynomial** of order n for f at x=a.

When a Taylor series or polynomial is centered at x=0, it is called a **Maclaurin series or polynomial,** respectively.

Example: Write the first order Taylor polynomial for $f(x)=\sqrt[4]{x}$ about x= 16 and use it to approximate $\sqrt[4]{15}$. Is this approximation an over or under-approximation? Explain your reasoning

A Taylor polynomial is a good approximation of a function for _________________________.

A Taylor polynomial over-approximates a value if the derivative for the ending term is______.

A Taylor polynomial under-approximates a value if the derivative for the ending term is______.

Example: Let j be a function having derivatives of all orders for $x > 0$. Selected values of j and its first four derivatives are indicated in the table below. The function j and these four derivative are increasing on the interval $1 \leq x \leq 4$.

Write the second-degree Taylor polynomial for j about $x = 1$ and use it to approximate $j(1.1)$. Is this approximation greater than $j(1.1)$? Explain your reasoning.

x	j(x)	j'(x)	j''(x)	j'''(x)
1	10	11	14	151/10
2	13	14	150/7	160/3
3	9	90/8	90/7	97/7
4	7	70/4	71/3	73/3

Practice: Use the table above to write the third-degree Taylor polynomial for j about $x = 2$ and use it to approximate $j(1.9)$. Is this approximation greater than $j(1.9)$? Explain your reasoning.

Practice: Use the table above to write the third-degree Taylor polynomial for j about $x = 3$. Is this Taylor polynomial only a good approximation of values of j near x=3? Explain.

Practice: Write the third order Taylor polynomial for $\ln(2-x)$ about x=1.

Practice: Selected values of a function f and its first four derivatives are shown in the table below. What is the approximation of the value of $f(2)$ obtained by using the third degree Taylor polynomial for f about x=1?

x	f(x)	f'(x)	f''(x)	f'''(x)
1	2	-3	6	-8

Example: The third degree Taylor polynomial for a function *f* about x=2 is:
$3+\frac{x-2}{3}+\frac{(x-2)^2}{27}+\frac{(x-2)^3}{243}$. What is the value of *f* '''(2)?

Example: Let $P(x)=2(x-3)^2-7(x-3)^3+5(x-3)^4$ be the fourth degree Taylor polynomial for the function *f* about x=3. What is the value of f '''(3)?

Practice: The third degree Taylor polynomial for a function *g* about x=5 is:
$-2+\frac{x-5}{2}+\frac{3(x-5)^2}{8}+\frac{9(x-5)^3}{32}$. What is the value of *g* ''(5)?

Practice: Let $P(x)=2-8x^2+3x^4-5x^6$ be the sixth degree Taylor polynomial for the function *f* about x=0. What is the value of $f^{(4)}(0)$?

Homework

1. a) Use the definition to find the Taylor series centered at $c = 1$ for $f(x) = \ln x$. Find the first four nonzero terms and then an expression for the nth term.

2. a) Use the definition to find the Maclaurin series for $f(x) = \sin 2x$. Find the first three nonzero terms and then an expression for the nth term.
   b) Can you think of a faster, more efficient method to obtain the series for $f(x) = \sin 2x$ other than using its derivatives? Explain.

3. Use the definition to find the Taylor series centered at $c = 1$ for $f(x) = \sqrt{x}$. Find only the first four nonzero terms.

4. a) Use the definition to find the Taylor series centered at $c = 2$ for $f(x) = 2^x$. Find the first four terms and then an expression for the nth term.

2003 AP® CALCULUS BC FREE-RESPONSE QUESTIONS (Form B)

5. The function f has a Taylor series about $x = 2$ that converges to $f(x)$ for all x in the interval of convergence. The nth derivative of f at $x = 2$ is given by $f^{(n)}(2) = \frac{(n+1)!}{3^n}$ for $n \geq 1$, and $f(2) = 1$.

   (a) Write the first four terms and the general term of the Taylor series for f about $x = 2$.

   (c) Let g be a function satisfying $g(2) = 3$ and $g'(x) = f(x)$ for all x. Write the first four terms and the general term of the Taylor series for g about $x = 2$.

Plus: Khan Sets